PurdueX: 416.2x
Probability: Distribution Models & Continuous Random Variables

Problem sets

Unit 7: Continuous Random Variables

STAT/MA 41600
Practice Problems: October 15, 2014

1. Consider a random variable X with density

$$f_X(x) = \begin{cases} \frac{1}{5}e^{-x/5} & \text{for } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

a. Find $P(3 \leq X \leq 5)$.

b. Find an expression for the CDF $F_X(x)$ of X.

c. Graph the CDF $F_X(x)$ of X.

2. Let X have density $f_X(x) = kx^2(1-x)^2$ for $0 \leq x \leq 1$, and $f_X(x) = 0$ otherwise, where k is constant.

a. Find the value of k.

b. Find $P(X \geq 3/4)$.

3. Assume X has constant density on the interval $[0, 25]$, and the density of X is 0 otherwise. Find $P(13.2 \leq X \leq 19.9)$.

4. Suppose X has CDF

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0, \\ x^4(5-4x) & \text{if } 0 \le x \le 1, \\ 1 & \text{if } x > 1. \end{cases}$$

a. Find $P(X > 1/2)$.

b. Find the density $f_X(x)$ of X.

5. Let X have density $f_X(x) = \frac{\sqrt{3(x+2)}}{6}$ for $-2 \leq x \leq 1$, and $f_X(x) = 0$ otherwise. Find the probability that X is positive.

STAT/MA 41600

Practice Problems: October 17, 2014

1. Consider a pair of random variables X, Y with constant joint density on the triangle with vertices at $(0, 0)$, $(3, 0)$, and $(0, 3)$. Find $P(X + Y > 2)$.

2. Consider a pair of random variables X, Y with constant joint density on the quadrilateral with vertices $(0, 0)$, $(2, 0)$, $(2, 6)$, $(0, 12)$. Find $P(Y \geq 3X)$.

3. Let X, Y have joint density $f_{X,Y}(x, y) = 14e^{-2x-7y}$ for $x > 0$ and $y > 0$; and $f_{X,Y}(x, y) = 0$ otherwise. Find $P(X > Y)$.

4. Suppose X, Y has joint density

$$f_{X,Y}(x,y) = \begin{cases} 1/16 & \text{if } -2 \leq x \leq 2 \text{ and } -2 \leq y \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Find $P(|X - Y| \leq 1)$.

5. Suppose X, Y has joint density

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{9}(3-x)(2-y) & \text{if } 0 \le x \le 3 \text{ and } 0 \le y \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Find $P(Y > X)$.

STAT/MA 41600
Practice Problems: October 20, 2014

1. Consider a pair of random variables X, Y with constant joint density on the triangle with vertices at $(0,0)$, $(3,0)$, and $(0,3)$.

a. Are X and Y independent? Why or why not?

b. Find the density $f_X(x)$ of X.

c. Find the density $f_Y(y)$ of Y.

2. Consider a pair of random variables X, Y with constant joint density on the quadrilateral with vertices $(0,0)$, $(2,0)$, $(2,6)$, $(0,12)$.

a. Are X and Y independent? Why or why not?

b. Find the density $f_X(x)$ of X.

c. Find the density $f_Y(y)$ of Y.

3. Let X, Y have joint density $f_{X,Y}(x,y) = 14e^{-2x-7y}$ for $x > 0$ and $y > 0$; and $f_{X,Y}(x,y) = 0$ otherwise.

a. Are X and Y independent? Why or why not?

b. Find the density $f_X(x)$ of X.

c. Find the density $f_Y(y)$ of Y.

4. Suppose X, Y has joint density

$$f_{X,Y}(x, y) = \begin{cases} 1/16 & \text{if } -2 \le x \le 2 \text{ and } -2 \le y \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

a. Are X and Y independent? Why or why not?

b. Find the density $f_X(x)$ of X.

c. Find the density $f_Y(y)$ of Y.

5. Suppose X, Y has joint density

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{9}(3-x)(2-y) & \text{if } 0 \le x \le 3 \text{ and } 0 \le y \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

a. Are X and Y independent? Why or why not?

b. Find the density $f_X(x)$ of X.

c. Find the density $f_Y(y)$ of Y.

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Probability: Distribution Models & Continuous Random Variables

Problem sets

Unit 8: Conditional Distributions and Expected Values

STAT/MA 41600
Practice Problems: October 22, 2014

1. Consider a pair of random variables X, Y with constant joint density on the triangle with vertices at $(0, 0)$, $(3, 0)$, and $(0, 3)$.
a. For $0 \le y \le 3$, find the conditional density $f_{X|Y}(x \mid y)$ of X, given $Y = y$.

b. Find the conditional probability that $X \le 1$, given $Y = 1$. I.e., find $P(X \le 1 \mid Y = 1)$.

c. Find the conditional probability that $X \le 1$, given $Y \le 1$. I.e., find $P(X \le 1 \mid Y \le 1)$.

2. Consider a pair of random variables X, Y with constant joint density on the quadrilateral with vertices $(0,0)$, $(2,0)$, $(2,6)$, $(0,12)$.
a. For $0 \leq y \leq 6$, find the conditional density $f_{X|Y}(x \mid y)$ of X, given $Y = y$.

b. For $6 \leq y \leq 12$, find the conditional density $f_{X|Y}(x \mid y)$ of X, given $Y = y$.

c. Find the conditional probability that $X \leq 1$, given $3 \leq Y \leq 9$.
I.e., find $P(X \leq 1 \mid 3 \leq Y \leq 9)$.

3. Let X, Y have joint density $f_{X,Y}(x, y) = 14e^{-2x-7y}$ for $x > 0$ and $y > 0$; and $f_{X,Y}(x, y) = 0$ otherwise.

a. For $y > 0$, find the conditional density $f_{X|Y}(x \mid y)$ of X, given $Y = y$.

b. Find the conditional probability that $X \geq 1$, given $Y = 3$. I.e., find $P(X \geq 1 \mid Y = 3)$.

c. Find the conditional probability that $Y \leq 1/5$, given $X = 2.7$.
I.e., find $P(Y \leq 1/5 \mid X = 2.7)$.

4. Let X, Y have joint density $f_{X,Y}(x,y) = 18e^{-2x-7y}$ for $0 < y < x$; and $f_{X,Y}(x,y) = 0$ otherwise.

a. For $y > 0$, find the conditional density $f_{X|Y}(x \mid y)$ of X, given $Y = y$.

b. For $x > 0$, find the conditional density $f_{Y|X}(y \mid x)$ of Y, given $X = x$.

5. Suppose X, Y has joint density

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{9}(3-x)(2-y) & \text{if } 0 \le x \le 3 \text{ and } 0 \le y \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

a. For $0 \le y \le 2$, find the conditional density $f_{X|Y}(x \mid y)$ of X, given $Y = y$.

b. Find the conditional probability that $X \le 2$, given $Y = 3/2$.
I.e., find $P(X \le 2 \mid Y = 3/2)$.

c. Find the conditional probability that $Y \ge 1$, given $X = 5/4$.
I.e., find $P(Y \ge 1 \mid X = 5/4)$.

STAT/MA 41600
Practice Problems: October 24, 2014

1. Consider a pair of random variables X, Y with constant joint density on the triangle with vertices at $(0,0)$, $(3,0)$, and $(0,3)$.

Find the expected value $\mathbb{E}(X)$. (Notice that, by symmetry, $\mathbb{E}(Y)$ is just the same!)

2. Consider a pair of random variables X, Y with constant joint density on the quadrilateral with vertices $(0,0)$, $(2,0)$, $(2,6)$, $(0,12)$.

a. Find the expected value $\mathbb{E}(X)$.

b. Find the expected value $\mathbb{E}(Y)$.

3. Let X, Y have joint density $f_{X,Y}(x, y) = 14e^{-2x-7y}$ for $x > 0$ and $y > 0$; and $f_{X,Y}(x, y) = 0$ otherwise.

a. Find the expected value $\mathbb{E}(X)$.

b. Find the expected value $\mathbb{E}(Y)$.

4. Let X, Y have joint density $f_{X,Y}(x, y) = 18e^{-2x-7y}$ for $0 < y < x$; and $f_{X,Y}(x, y) = 0$ otherwise.

a. Find the expected value $\mathbb{E}(X)$.

b. Find the expected value $\mathbb{E}(Y)$.

5. Suppose X, Y has joint density

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{9}(3-x)(2-y) & \text{if } 0 \leq x \leq 3 \text{ and } 0 \leq y \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

a. Find the expected value $\mathbb{E}(X)$.

b. Find the expected value $\mathbb{E}(Y)$.

STAT/MA 41600
Practice Problems: October 27, 2014

1. Consider a pair of random variables X, Y with constant joint density on the triangle with vertices at $(0, 0)$, $(3, 0)$, and $(0, 3)$.

a. Find the expected value of the sum of X and Y, i.e., find $\mathbb{E}(X + Y)$.

b. Find the variance of X, i.e., find $\mathrm{Var}(X)$.

2. Consider a pair of random variables X, Y with constant joint density on the quadrilateral with vertices $(0,0)$, $(2,0)$, $(2,6)$, $(0,12)$.

a. Find the variance of X, i.e., find $\operatorname{Var} X$.

b. Find the variance of Y, i.e., find $\operatorname{Var} Y$.

3. Let X, Y have joint density $f_{X,Y}(x, y) = 14e^{-2x-7y}$ for $x > 0$ and $y > 0$; and $f_{X,Y}(x, y) = 0$ otherwise.

Find the variance of the sum of X and Y, i.e., find $\mathrm{Var}(X + Y)$.

4. Let X, Y have joint density $f_{X,Y}(x,y) = 18e^{-2x-7y}$ for $0 < y < x$; and $f_{X,Y}(x,y) = 0$ otherwise.

Find the variance of Y.

5. Suppose X, Y has joint density

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{9}(3-x)(2-y) & \text{if } 0 \le x \le 3 \text{ and } 0 \le y \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Find the expected value of $X^2 + Y^3$, i.e., find $\mathbb{E}(X^2 + Y^3)$.

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Probability: Distribution Models & Continuous Random Variables

Problem sets

Unit 9: Models of Continuous Random Variables

STAT/MA 41600
Practice Problems: October 29, 2014

1. Suppose that X has CDF

$$F_X(x) = \begin{cases} 0 & x < 2, \\ \frac{x-2}{4} & 2 \le x \le 6, \\ 1 & x > 6. \end{cases}$$

a. Find $P(X \le 4.5)$.

b. Find $P(3.09 \le X \le 4.39)$.

c. Find $P(X \ge 3.7)$.

2. Assume that the amount of soda in a can is uniformly distributed between 11.93 ounces and 12.02 ounces.

a. What is the probability that it has more than the stated quantity that is printed on the can, which is 12 ounces?

b. What is the standard deviation of the amount (in ounces) of soda in the can? (Remember that the standard deviation is the square root of the variance.)

3. Suppose that, when you buy gas at the gas station, the price is uniformly distributed between \$4.30 and \$4.50 per gallon. You plan to buy 12 gallons of gasoline, plus a candy bar for an extra \$1.00. (Assume that there is no tax on your purchase.)

a. Find the expected value of the cost of your purchase.

b. Find the variance of the cost of your purchase.

4. Let X, Y, Z be independent and uniformly distributed on the interval $[0, 10]$. Find the probability that Y is the middle value, i.e., find $P(X < Y < Z \text{ or } Z < Y < X)$.

5. Suppose X, Y have constant joint density on the triangle with vertices at $(0,0)$, $(3,0)$, and $(0,3)$.

Find $\mathbb{E}(\min(X, Y))$.

STAT/MA 41600
Practice Problems: October 31, 2014

1. The waiting time until the phone rings is exponential, with an average waiting time of 30 minutes.

a. What is the probability that nobody calls during the next hour?

b. What is the standard deviation (in minutes) of time until the next call?

2. A chef working in a kitchen believes that the waiting time until the next dessert order is exponential, with an average of 3 minutes. The waiting time until the next appetizer order is also exponential, with an average of 2 minutes. These waiting times are independent.

Find the probability that the next dessert is ordered before the next appetizer.

3. Suppose that the times until Hector, Ivan, and Jacob's pizza arrives are independent exponential random variables, each with average of 20 minutes. Find the probability that none of the waiting times exceed 20 minutes. I.e., find $P(\max(X, Y, Z) \leq 20)$.

4. Suppose that, when an airplane waits on the runway, the company must pay each customer a fee if the waiting time exceeds 3 hours. Suppose that an airplane with 72 passengers waits an exponential amount of time on the runway, with average 1.5 hours. If the waiting time X, in hours, is bigger than 3, then the company pays each customer $(100)(X - 3)$ dollars (otherwise, the company pays nothing). What is the amount that the company expects to pay for the 72 customers on the airplane altogether? (Of course their waiting times are all the same.)

5. Let X be uniform on $[0, 10]$. Let Y be exponential with $\mathbb{E}(Y) = 5$. Find $P(X < Y)$.

STAT/MA 41600

Practice Problems: November 5, 2014

1. Let X_1, X_2, X_3 be independent exponential waiting times, each with an average of 30 minutes. Let $Y = X_1 + X_2 + X_3$.

a. What is the average (in minutes) of Y?

b. What is the standard deviation (in minutes) of Y?

2. A chef working in a kitchen believes that the waiting time until the next dessert order is exponential, with an average of 3 minutes. The times between dessert orders are assumed to be independent exponentials, also with 3 minutes on average. Let Y be the time until the next dessert order, and let Z be the subsequent time (afterwards) until the following dessert order.

[E.g., if it is 12 noon right now, and the next order arrives at 12:04 PM, and the order after that arrives at 12:11 PM, then $Y = 4$ and $Z = 7$.]

Let $X = Y + Z$. Find the density of X.

3. Suppose that the times until Hector, Ivan, and Jacob's pizzas arrive are independent exponential random variables, each with average of 20 minutes. Let X be the *sum* of the times that they spend waiting, i.e., Hector's time plus Ivan's time plus Jacob's time. Find the variance of X.

4. [Question about Exponential random variables.]

Let X be exponential with expected value 3. Let Y be another random variable that depends on X as follows: if $X > 5$, then $Y = X - 5$; otherwise, $Y = 0$.

a. Find the expected value of Y.

b. Find the variance of Y.

5. [Question about Exponential random variables.]

Suppose that Michelle, Nancy, and Olivia each are waiting for their husbands to appear. Their waiting times are assumed to be independent exponentials, and they each expect to wait 5 minutes. Let X denote the time until the very first husband appears.

What is the expected value of X? [Hint: Since X is the minimum of three independent exponential random variables, then X is exponential.]

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Probability: Distribution Models & Continuous Random Variables

Problem sets

Unit 10: Normal Distribution and Central Limit Theorem (CLT)

STAT/MA 41600
Practice Problems: November 10, 2014

1. Assume X is normally distributed with $\mathbb{E}(X) = 4.2$ and $\operatorname{Var} X = 50.41$.

a. Find $P(X \leq 10)$.

b. Find $P(X \leq 0)$.

c. Find $P(0 \leq X \leq 10)$.

2. In a certain class, the student scores are approximately normally distributed, with mean 72.5% and standard deviation 6.9%. What percent of students score above 70%?

3. Find the value of a so that, if Z is a standard normal random variable, then

$$P(a \leq Z \leq .54) = 0.3898.$$

4. Assume that the height of an American female is normal with expected value $\mu = 64$ and standard deviation $\sigma = 12.8$.

a. What is the probability that an American female's height is 66 inches or taller?

b. The heights of 10 American females are measured (in inches). Assume that their heights are independent. What is the expected number of these females who have height 66 inches or taller?

5. The quantity of sugar X (measured in grams) in a randomly-selected piece of candy is normally distributed, with expected value $\mathbb{E}(X) = \mu = 22$ and variance $\text{Var}(X) = \sigma^2 = 8$.

For which value of x is $P(X \leq x) = 0.1492$? In other words, find the quantity x of sugar, so that exactly 14.92% of the candy has less than x grams of sugar.

STAT/MA 41600
Practice Problems: November 12, 2014

1a. If X_1, X_2, X_3, X_4, X_5 are independent normal random variables that each have average 8.2 and variance 32.49, then the sum of the X_j's is normal. Thus, if we divide the sum of the X_j's by 5, we get the average of the X_j's, which is normal too: $Y = \frac{X_1+X_2+X_3+X_4+X_5}{5}$.

Find the expected value $\mathbb{E}(Y)$, and find the variance $\text{Var}(Y)$.

1b. If $X_1, X_2, \ldots, X_n$ are independent normal random variables that each have average μ and variance σ^2, then the sum of the X_j's is normal. Thus, if we divide the sum of the X_j's by n, we get the average of the X_j's, which is normal too: $Y = \frac{X_1+X_2+\cdots+X_n}{n}$.

Find the expected value $\mathbb{E}(Y)$, and find the variance $\text{Var}(Y)$.

2. Assume that the quantity of money in a randomly-selected checking account is normal with mean \$1325 and standard deviation \$25. Also assume that the amounts in different accounts of different people are independent. Let X be the sum of the money contained (altogether) in three randomly-chosen people's accounts. Find the probability that X exceeds \$4000.

3. The time that it takes a random person to get a haircut is normally distributed, with an average of 23.8 minutes and a standard deviation of 5 minutes. Assume that different people have independent times of getting their hair cut. Find the probability that, if there are four customers in a row (with no gaps in between), they will all be finished getting their hair cut in 1.5 hours (altogether) or less.

4. Assume that the height of an American female is normal with expected value $\mu = 64$ inches and standard deviation $\sigma = 12.8$ inches. Also assume that different women have independent heights.

Measure the heights $X_1, \ldots, X_{10}$ of ten women. Let $Y = \frac{X_1+X_2+\cdots+X_{10}}{10}$ denote their average height. Find the probability that Y exceeds 60 inches.

5. The quantity of sugar X (measured in grams) in a randomly-selected piece of candy is normally distributed, with expected value $\mathbb{E}(X) = \mu = 22$ and variance $\text{Var}(X) = \sigma^2 = 8$. Assume that different pieces of candy have independent quantities of sugar. Find the probability that, in a handful containing 7 pieces of candy, there are 150 or more grams of sugar.

STAT/MA 41600

Practice Problems: November 14, 2014

1. Each of the 30 students in a class each orders a package. They assume that the waiting time (measured in days) for the packages are independent exponential random variables, with average waiting time of 1/2 for each package.

What is the approximate probability that the total waiting time exceeds 14 days?

2. When the students in question #1 eventually receive their packages, sometimes they are happy with the items they ordered, and sometimes they are not. Suppose that a student is happy with her/his own package with probability 0.60, independent of the happiness/unhappiness of the other students.

What is an estimate for the probability that 20 or more of the students are happy with their packages?

3. In planning for an event, the planner estimates that nobody will be on time, but nobody will be more than 10 minutes late. So he estimates that the time (in minutes) a given person will be late has density

$$f_X(x) = \frac{(10-x)^3}{2500}, \qquad \text{for } 0 \leq x \leq 10,$$

and $f_X(x) = 0$ otherwise.

a. Find the expected value and variance of X. Hint: It might be helpful to use the u-substitution $u = 10 - x$.

b. Estimate the probability that, among a group of 200 attendees who behave independently and follow the behavior described above, the total sum of their delay in arriving is more than 420 minutes, i.e., 7 hours.

4. Suppose that 100 marathon runners each complete a marathon in 3.5 hours, on average, with standard deviation 0.5 hours. Estimate the probability that the sum of their completion times is between 348 and 352 hours.

5. Barbara is an inspector for a water bottling company. She notices that the amount of water in each bottle has an average of 0.99 liters, and a standard deviation of 0.03 liters. She measures the quantities $X_1, \ldots, X_{12}$ in twelve independent bottles, and computes the average, Y, in these 12 bottles, i.e., $Y = \frac{X_1 + \cdots + X_{12}}{12}$. Estimate the probability that $Y \geq 1$, i.e., that the average amount of water in the twelve bottles exceeds 1 liter.

STAT/MA 41600

Practice Problems #2: November 14, 2014

1. There are 2000 flights that arrive at the Denver airport each day, 70% of which are on time. What is the approximate probability that strictly more than 1420 of the flights are on time in one day? Assume that the flights' delays are relatively independent.

2. There is a big exam tonight, and all of the 400 students are invited to attend the help session. From past experience, the instructor finds that each student is 60% likely to attend the help session. If the students behave independently, find the probability that between 230 and 250 (inclusive) students attend the help session.

3. Bob is a professional crayon inspector. Each crayon he checks has a 5% chance of being broken. If he checks 12,000 crayons during a certain production run, what is the approximate probability that there are between 580 and 620 (inclusive) crayons that are broken?

4. If 6% of passengers are screened with an extra round of security at the airport, and Southwest has 8 flights with 180 passengers each, what is the approximate probability that 80 or more of them will receive this extra level of screening?

5. Jeff typically makes 80% of his field goals. Steve typically makes 60% of his field goals. If Jeff tries 120 times to get a field goal during a season, and Steve tries 164 times to get a field goal during a season, and their attempts are independent, approximate the probability that Jeff gets strictly more field goals than Steve.

6. Design your own problem and solution. Create your own problem about normal approximation to a Binomial random variable with a large parameter n. Design your problem in such a way that you would find it enjoyable and also interesting (i.e., not completely trivial) if you found this problem in a probability book. Please provide the answer for your problem too.

STAT/MA 41600
Practice Problems #3: November 14, 2014

1. Suppose that the number of Roseate Spoonbills (a very rare bird in Indiana) that fly overhead in 1 hour has a Poisson distribution with mean 2. Also suppose that the number of Roseate Spoonbills is independent from hour to hour (e.g., the number of birds between noon and 1 PM does not affect the number of birds between 1 PM and 2 PM, etc.).

During 40 hours of observation, what is the approximate probability that 75 or more Roseate Spoonbills are seen?

2. As you know, Dr. Ward likes to be extremely careful with his writing, but alas we are all human, so tiny errors do occasionally appear. As in Problem Set 17, Question 2, suppose that Dr. Ward has an average of only 0.04 errors per page when writing (i.e., approximately 1 error every 25 pages).

If Dr. Ward will write 6000 pages of text during his entire life as an author, what is the approximate probability that he will make strictly less than 230 errors altogether in his lifetime of publications?

3. Bob is a professional crayon inspector. On average, Bob can check about 295 crayons per hour. What is the approximate probability that he can check all 12,000 crayons from a certain production run, during his 40-hour work week (without needing to request overtime)?

4. Customers arrive at a round-the-clock gas station, with an average rate of 8 per hour. During a full week (which is 168 hours), what is the approximate probability that between 1300 and 1400 customers (inclusive) will arrive at the gas station?

5. Dr. Ward's wife writes novels and short stories. Suppose that she plans to write 10,000 pages during her lifetime. She has an average of only 0.025 errors per page when writing (i.e., approximately 1 error every 40 pages).

What is the approximate probability that Dr. Ward's wife makes strictly fewer errors than him, during their entire lives?

PurdueX: 416.2x
Probability: Distribution Models & Continuous Random Variables

Problem sets

Unit 11: Covariance, Conditional Expectation, Markov and Chebychev Inequalities

STAT/MA 41600
Practice Problems: November 24, 2014

1. Consider a tray with 8 lemonades and 3 raspberry juices. Alice and Bob each take 1 drink from the tray, without replacement. Assume that all of their choices are equally likely. Let X be the number of lemonades that Alice and Bob get. (Note: X is either 0, 1, or 2.) Find the variance of X.

2. In question #1, let X_1 and X_2 indicate (respectively) if Alice and Bob (respectively) get lemonade. In other words, $X_1 = 1$ if Alice gets lemonade, or $X_1 = 0$ otherwise; and $X_2 = 1$ if Bob gets lemonade, or $X_2 = 0$ otherwise.

Find the correlation $\rho(X_1, X_2)$ between X_1 and X_2.

3a. Suppose that X is a continuous random variable that is Uniformly distributed on $[10, 14]$, and suppose $Y = 2X + 2$. Find $\text{Cov}(X, Y)$, i.e., the covariance of X and Y.

b. Find the correlation $\rho(X, Y)$ of X and Y.

4a. Suppose that X is a continuous random variable that is Uniformly distributed on $[3, 6]$, and suppose $Y = (X-1)(X+1) = X^2 - 1$. Find $\text{Cov}(X, Y)$, i.e., the covariance of X and Y.

b. Find the correlation $\rho(X, Y)$ of X and Y.

5. Roll two 4-sided dice (*not 6-sided dice*). Let X be the minimum value, and let Y be the maximum value. Find the covariance of X and Y.

STAT/MA 41600

Practice Problems: December 1, 2014

1. Let X and Y have a joint uniform distribution on the triangle with corners at $(0, 2)$, $(2, 0)$, and the origin. Find $\mathbb{E}(Y \mid X = 1/2)$.

2. Roll two 6-sided dice. Let X denote the minimum value that appears, and let Y denote the maximum value that appears.

a. Find $\mathbb{E}(Y \mid X = 3)$.

b. Find $\mathbb{E}(X + Y \mid X = 3)$. [Hint: Using (a)'s answer, you can solve (b) in one line!]

3. Let X_1 and X_2 be independent exponential random variables, each with mean 1. Let $Y = X_1 + X_2$. Find $\mathbb{E}(X_1 \mid Y = 3)$.

4. Consider a tray with 8 lemonades and 3 raspberry juices. Alice and Bob each take 1 drink from the tray, without replacement. Assume that all of their choices are equally likely. Let X_1 and X_2 indicate (respectively) if Alice and Bob (respectively) get lemonade. In other words, $X_1 = 1$ if Alice gets lemonade, or $X_1 = 0$ otherwise; and $X_2 = 1$ if Bob gets lemonade, or $X_2 = 0$ otherwise.

a. Find $\mathbb{E}(X_1 \mid X_2 = 1)$.

b. Find $\mathbb{E}(X_1 \mid X_2 = 0)$.

5. Sally and David each pick 10 flowers from the case without paying attention to what type of flowers they are picking. There are a large quantity of flowers available, 20% of which are roses. Let X be the number of roses that Sally picks, and let Y be the number of roses that the couple picks altogether. Find the number of roses that we expect Sally to pick if the total number of roses picked is $Y = 12$.

STAT/MA 41600
Practice Problems: December 3, 2014

1a. The average amount of time that a student spends studying for a final exam is 5 hours. Find an upper-bound on the probability that a student spends 7 or more hours studying for a final exam.

1b. Now also assume that the standard deviation of the study time for a final exam is 1.25 hours. Find a lower-bound on the probability that the time spent studying is between 3 to 7 hours.

2. Henry caught a cold recently and therefore he has been sneezing a lot. His expected waiting time between sneezes is 35 seconds. The standard deviation of the waiting time between his sneezes is 1.5 seconds. Find a bound on the probability that the time between two consecutive sneezes is between 30 and 40 seconds.

3. In a study on eating habits, a particular participant averages 750 cm^3 of food per meal.

a. It is extraordinarily rare for this participant to eat more than 1000 cm^3 of food at once. Find a bound on the probability of such an event.

b. If the standard deviation of a meal size is 100 cm^3, the find a bound on the event that the meal is either too much food, i.e., more than 1000 cm^3, or an insufficient amount of food, namely, less than 500 cm^3.

4. (Review) People shopping at the grocery store are interviewed to see whether or not they enjoy artichokes. Only 11% of people like artichokes.

a. How many people does the interviewer expect to meet until finding the 25th person who likes artichokes?

b. What is the variance of the number of people he meets, to find this 25th person who likes artichokes?

5. (Review) On a Monday evening, a student begins to wait for the telephone to ring. Let X_1 be the time until the telephone rings the first time. The student then picks up the phone, immediately recognizes he does not want to talk to the person, hangs up the phone, and begins to wait again for the second call. Let X_2 denote the waiting time after the first call, until the phone rings a second time. Assume X_1, X_2 are independent exponential random variables, each with mean 10 (minutes). Let $Y = X_1 + X_2$ be the total waiting time.

a. What kind of random variable is Y?

b. Find the expected value of Y.

c. Find the variance of Y.

d. Find the probability that Y exceeds 12 minutes, i.e., find $P(Y > 12)$.

PurdueX: 416.2x
Probability: Distribution Models & Continuous Random Variables

Problem sets

Unit 12: Order Statistics, Moment Generating Functions, Transformation of RVs

STAT/MA 41600

Practice Problems: December 5, 2014

1. Consider X_1, X_2, X_3, X_4 which are independent and uniformly distributed on $[0, 20]$.

a. Find the density of the first order statistic, i.e., find $f_{X_{(1)}}(x_1)$.

b. Find the density of the second order statistic, i.e., find $f_{X_{(2)}}(x_2)$.

2. Same setup as Question #1.
a. Find the expected value of the first order statistic, i.e., find $\mathbb{E}(X_{(1)})$.

b. Find the expected value of the second order statistic, i.e., find $\mathbb{E}(X_{(2)})$.

3. Let X_1 and X_2 be the waiting times for Alice and Bob until their respective phones ring. Assume that X_1, X_2 are independent exponentials, each with mean 10.

a. Find the density of the first order statistic, $X_{(1)}$, i.e., find $f_{X_{(1)}}(x_1)$.

b. Find the density of the second order statistic, $X_{(2)}$, i.e., find $f_{X_{(2)}}(x_2)$.

4. Same setup as Question #3.
a. Find the expected value of the first order statistic, i.e., find $\mathbb{E}(X_{(1)})$.

b. Find the expected value of the second order statistic, i.e., find $\mathbb{E}(X_{(2)})$.

5. Let X_1, X_2 be independent, identically distributed, each with density $f_X(x) = 6(x - x^2)$ for $0 < x < 1$, and $f_X(x) = 0$ otherwise.

a. Find the density of the first order statistic, $X_{(1)}$.

b. Find the expected value of the first order statistic, i.e., find $\mathbb{E}(X_{(1)})$.

STAT/MA 41600
Practice Problems: December 10, 2014

1. Jim cuts wood planks of length X for customers, where X is uniformly distributed between 10 and 14 feet. The price of a piece of wood is \$2 per foot, plus a flat-rate surcharge of \$2 for Jim's services. So $Y = 2X + 2$ is the amount he charges for a piece of wood.

a. Find the density $f_Y(y)$. Be sure to specify the interval where $f_Y(y)$ is nonzero.

b. Using the density $f_Y(y)$ of Y, find the probability that Y exceeds \$28.

c. Check your answer: Using the density $f_X(x)$ of X, find the probability that $Y = 2X + 2$ exceeds \$28.

2. Let X be the price of a CD during a "lightning sale" on Cyber Monday. The total purchase price (in dollars) is $Y = 1.07X + 3.99$, since there is 7% tax and \$3.99 shipping. Suppose that X is uniform on the interval $[4, 9]$.

a. Find the density $f_Y(y)$. Be sure to specify the interval where $f_Y(y)$ is nonzero.

b. Find the expected purchase price (with tax and shipping) by $\mathbb{E}(Y) = \int y\, f_Y(y)\, dy$.

c. Check your answer: Find the expected purchase price (with tax and shipping) by $\mathbb{E}(1.07X + 3.99) = \int_4^9 (1.07x + 3.99)\, f_X(x)\, dx$.

3. Let X be a random variable that is uniform on $[3, 6]$.

a. If we make a box with area $Y = (X - 1)(X + 1)$, what is the CDF of Y? [Hint: Note that $(2)(4) \leq Y \leq (5)(7)$, i.e., $8 \leq Y \leq 35$.]

b. What is the density $f_Y(y)$ of the area Y of the box?

c. Use the density of Y to get the expected area by integrating, i.e., $\mathbb{E}(Y) = \int_8^{35} y\, f_Y(y)\, dy$.

d. Check your answer: Integrate with respect X to get the same expected area, i.e., $\mathbb{E}((X - 1)(X + 1)) = \int_3^6 (x - 1)(x + 1) f_X(x)\, dx$.

4. Let X and Y have a joint uniform distribution on the triangle with corners at $(0, 2)$, $(2, 0)$, and the origin. Find the covariance of X and Y.

5. There are 20 chairs in a circle, and 10 pairs of married individuals. Assume all seating arrangements are equally likely. Let X be the number of couples sitting together, i.e., let X be the number of men who are sitting next to their own wives. Find the variance of X.

Hint: Write $X = X_1 + \cdots + X_{10}$, where X_j indicates whether the jth couple sits together. Then

$$\mathrm{Var}(X) = \mathrm{Var}(X_1 + \cdots + X_{10}) = \sum_{j=1}^{10} \mathrm{Var}(X_j) + 2 \sum\sum_{1 \le i < j \le 10} \mathrm{Cov}(X_i, X_j).$$

[There are 10 terms of the first type and 90 terms of the second type.]