PurdueX: 416.2x
Probability: Distribution Models & Continuous Random Variables

Solution to the problem sets

Unit 7: Continuous Random Variables

STAT/MA 41600
Practice Problems: October 15, 2014
Solutions by Mark Daniel Ward

1.

a. We compute $P(3 \leq X \leq 5) = \int_3^5 \frac{1}{5} e^{-x/5}\,dx = -e^{-x/5}\big|_{x=3}^{5} = e^{-3/5} - e^{-1} = 0.1809$.

b. For $a \leq 0$, $F_X(a) = 0$ since the density is 0 for $x < a$. For $a > 0$, $F_X(a) = \int_{-\infty}^{a} f_X(x)\,dx = \int_0^a \frac{1}{5} e^{-x/5}\,dx = -e^{-x/5}\big|_{x=0}^{a} = 1 - e^{-a/5}$. Thus

$$F_X(x) = \begin{cases} 1 - e^{-x/5} & \text{for } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

c.

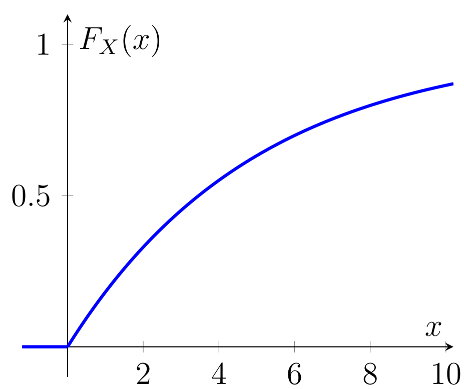

Figure 1: The CDF $F_X(x) = 1 - e^{-x/5}$ of X.

2.

a. We compute

$$1 = \int_0^1 kx^2(1-x)^2\,dx = k\int_0^1 (x^2 - 2x^3 + x^4)\,dx = k\left(\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5}\right)\Bigg|_{x=0}^{1} = k\left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right) = k/30$$

and thus $k = 30$.

b. We compute

$$\int_{3/4}^1 30x^2(1-x)^2\,dx = 30\int_{3/4}^1 (x^2 - 2x^3 + x^4)\,dx = 30\left(\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5}\right)\Bigg|_{x=3/4}^{1} = 53/512 = 0.1035.$$

As an alternative method, we could have used u-substitution with $u = 1 - x$ at the start, so that one of the limits of integration becomes 0. We get

$$\int_0^{1/4} 30(1-u)^2u^2\,du = 30\int_0^{1/4} (u^2 - 2u^3 + u^4)\,dx = 30\left(\frac{u^3}{3} - \frac{2u^4}{4} + \frac{u^5}{5}\right)\Bigg|_{x=0}^{1/4} = 53/512 = 0.1035.$$

3. We have $f_X(x) = k$ for $0 \le x \le 25$, and thus $1 = \int_0^{25} k\,dx = kx|_{x=0}^{25} = 25k$, so $k = 1/25$. Thus $f_X(x) = 1/25$ for $0 \le x \le 25$, and $f_X(x) = 0$ otherwise.

So $P(13.2 \le X \le 19.9) = \int_{13.2}^{19.9} 1/25\,dx = x/25|_{x=13.2}^{19.9} = \frac{19.9-13.2}{25} = 0.268$.

4.

a. Find $P(X > 1/2)$.

We have $P(X > 1/2) = 1 - P(X \le 1/2) = 1 - F_X(1/2) = 1 - (1/2)^4(5 - 4/2) = 1 - (1/16)(6/2) = 1 - 3/16 = 13/16$.

b. The density is the derivative of the CDF. Thus, for $x < 0$ and for $x > 1$, the density is $f_X(x) = 0$. For $0 \le x \le 1$, the density is $f_X(x) = \frac{d}{dx}(x^4(5-4x)) = 4x^3(5-4x) + x^4(-4) = 20x^3 - 20x^4 = 20x^3(1-x)$. So the density of X is

$$f_X(x) = \begin{cases} 20x^3(1-x) & \text{if } 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

5. We use u-substitution with $u = x + 2$, to compute $P(X > 0) = \int_0^1 \frac{\sqrt{3(x+2)}}{6}\,dx = \int_2^3 \frac{\sqrt{3u}}{6}\,du = \frac{\sqrt{3}}{6}\frac{u^{3/2}}{3/2}\Big|_{u=2}^{3} = \frac{\sqrt{3}}{9}\left(3^{3/2} - 2^{3/2}\right) = 1 - \frac{2\sqrt{6}}{9} = 0.4557$.

STAT/MA 41600
Practice Problems: October 17, 2014
Solutions by Mark Daniel Ward

1. The joint density is constant on a region of area $(3)(3)/2 = 9/2$. So the joint density $f_X(x)$ is $2/9$ on the triangle, and 0 otherwise.
Method #1: We integrate $2/9$ over the region, which is shown in Figure 1(a), and we get

$$\begin{aligned}
\int_0^2 \int_{2-x}^{3-x} \frac{2}{9}\,dy\,dx + \int_2^3 \int_0^{3-x} \frac{2}{9}\,dy\,dx &= \int_0^2 \frac{2}{9}y\Big|_{y=2-x}^{3-x}\,dx + \int_2^3 \frac{2}{9}y\Big|_{y=0}^{3-x}\,dx \\
&= \int_0^2 \frac{2}{9}\,dx + \int_2^3 \frac{2}{9}(3-x)\,dx \\
&= \frac{2}{9}x\Big|_{x=0}^{2} + \frac{2}{9}\left(3x - \frac{x^2}{2}\right)\Big|_{x=2}^{3} \\
&= 4/9 + 1/9 \\
&= 5/9
\end{aligned}$$

Method #2: We integrate $2/9$ over the complementary region, which is shown in Figure 1(b), and we get

$$\begin{aligned}
1 - \int_0^2 \int_0^{2-x} \frac{2}{9}\,dy\,dx &= 1 - \int_0^2 \frac{2}{9}y\Big|_{y=0}^{2-x}\,dx \\
&= 1 - \int_0^2 \frac{2}{9}(2-x)\,dx \\
&= 1 - \frac{2}{9}\left(2x - \frac{x^2}{2}\right)\Big|_{x=0}^{2} \\
&= 1 - (2/9)(4-2) \\
&= 1 - 4/9 \\
&= 5/9
\end{aligned}$$

Method #3: Actually we don't need to integrate a constant density. We integrate the constant over a region, so the integral is the area of the shaded region (here, $5/2$; see Figure 1(a)) over the area of the whole region (here, $9/2$), so the probability is $\frac{5/2}{9/2} = 5/9$.

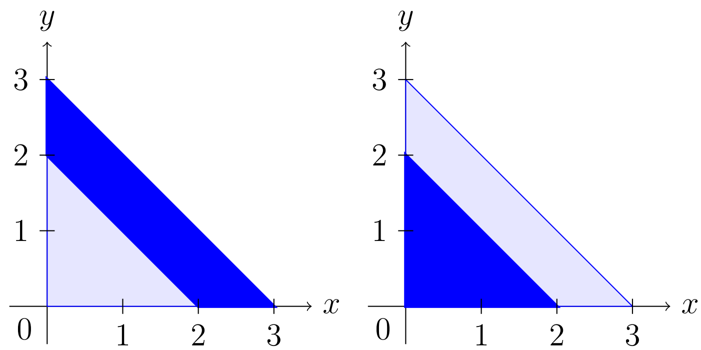

Figure 1: (a.) The region where $X+Y > 2$; (b.) the complementary region, where $X+Y < 2$.

2. The joint density is constant on a region of area 18. So the joint density $f_X(x)$ is 1/18 on the quadrilateral, and 0 otherwise.
Method #1: We integrate 1/18 over the region, which is shown in Figure 2(a), and we get

$$\int_0^2 \int_{3x}^{12-3x} \frac{1}{18}\,dy\,dx = \int_0^2 \frac{1}{18}y\Big|_{y=3x}^{12-3x}\,dx = \int_0^2 \frac{1}{18}(12-6x)\,dx = \frac{1}{18}\left(12x-3x^2\right)\Big|_{x=0}^{2} = 2/3.$$

Method #2: We integrate 1/18 over the complementary region, which is shown in Figure 2(b), and we get

$$1 - \int_0^2 \int_0^{3x} \frac{1}{18}\,dy\,dx = 1 - \int_0^2 \frac{1}{18}y\Big|_{y=0}^{3x}\,dx = 1 - \int_0^2 x/6\,dx = 1 - x^2/12\big|_{x=0}^{2} = 2/3.$$

Method #3: Actually we don't need to integrate a constant density. We integrate the constant over a region, so the integral is the area of the shaded region (here, 12; see Figure 2(a)) over the area of the whole region (here, 18), so the probability is 12/18 = 2/3.

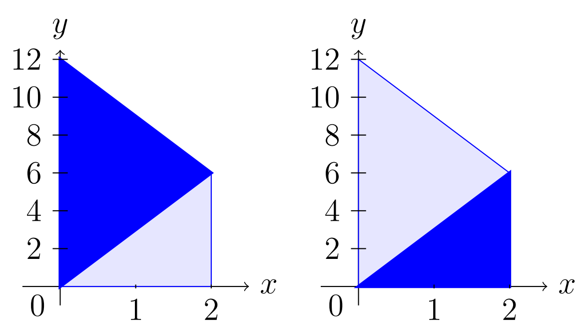

Figure 2: (a.) The region where $Y \geq 3X$; (b.) the complementary region, where $Y \leq 3X$.

3. We have two ways to setup the integral:

Method #1: We can integrate first over all x's (i.e., use integration with respect to x as the outer integral), and then fix x and integrate over all of the y's that are smaller than x, namely, $0 \le y \le x$, as shown in Figure 3.

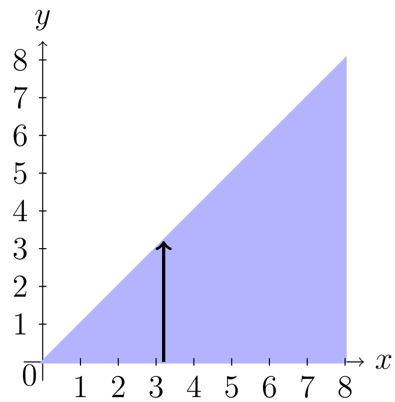

Figure 3: Setting up the integral for $P(X > Y)$, with x as the outer integral and y as the inner integral. Fixed value of x (here, for example $x = 3.2$), and y ranging from 0 to x.

Now we perform the joint integral, as specified in Figure 3, and we get

$$\begin{aligned} P(X > Y) &= \int_0^\infty \int_0^x 14e^{-2x-7y}\,dy\,dx \\ &= \int_0^\infty -2e^{-2x-7y}\Big|_{y=0}^{x}\,dx \\ &= \int_0^\infty (2e^{-2x} - 2e^{-9x})\,dx \\ &= (-e^{-2x} + (2/9)e^{-9x})\Big|_{x=0}^{\infty} \\ &= (1 - (2/9)) \\ &= 7/9 \end{aligned}$$

Method #2: We can integrate first over all y's (i.e., integrating with respect to y as the outer integral), and then fix y and integrate over all of the x's that are larger than y, namely, $y \le x$, as shown in Figure 4.

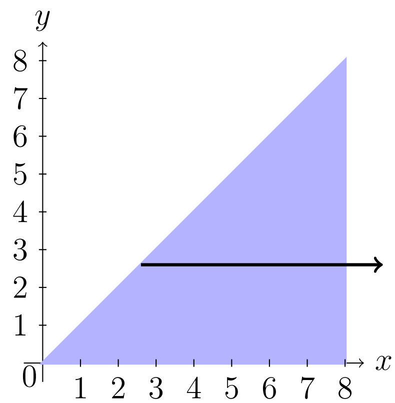

Figure 4: Setting up the integral for $P(X > Y)$, with y as the outer integral and x as the inner integral. Fixed value of y (here, for example $y = 2.6$), and x ranging from y to ∞.

Now we perform the joint integral, as specified in Figure 4, and we get

$$\begin{aligned} P(X > Y) &= \int_0^\infty \int_y^\infty 14e^{-2x-7y}\,dx\,dy \\ &= \int_0^\infty -7e^{-2x-7y}\Big|_{x=y}^{\infty}\,dy \\ &= \int_0^\infty 7e^{-9y}\,dy \\ &= -(7/9)e^{-9y}\Big|_{y=0}^{\infty} \\ &= 7/9 \end{aligned}$$

4. *Method #1:* We can integrate the joint density over the region where $|X - Y| \le 1$, which is shown in Figure 5. The desired probability is

$$\begin{aligned} &\int_{-2}^{-1}\int_{-2}^{x+1} 1/16\,dy\,dx + \int_{-1}^{1}\int_{x-1}^{x+1} 1/16\,dy\,dx + \int_{1}^{2}\int_{x-1}^{2} 1/16\,dy\,dx \\ &\quad= \int_{-2}^{-1} \frac{x+3}{16}\,dx + \int_{-1}^{1} \frac{2}{16}\,dx + \int_{1}^{2} \frac{3-x}{16}\,dx \\ &\quad= \frac{x^2/2+3x}{16}\Big|_{x=-2}^{-1} + \frac{2x}{16}\Big|_{x=-1}^{1} + \frac{3x-x^2/2}{16}\Big|_{x=1}^{2} \\ &= 3/32 + 4/16 + 3/32 \\ &= 14/32 \\ &= 7/16 \end{aligned}$$

Method #2: The desired region has area 7, and the entire region has area 16. Since the joint density is constant, it follows that $P(|X - Y| \le 1) = 7/16$.

Method #3: The complementary region has area 9, and the entire region has area 16. Since the joint density is constant, it follows that $P(|X - Y| \le 1) = 1 - 9/16 = 7/16$.

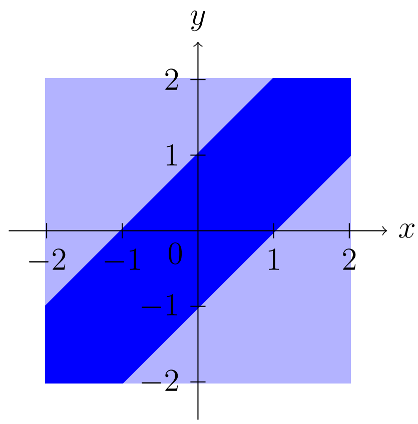

Figure 5: Setting up the integral for $P(|X - Y| \leq 1)$.

5. The region is shown in Figure 6.

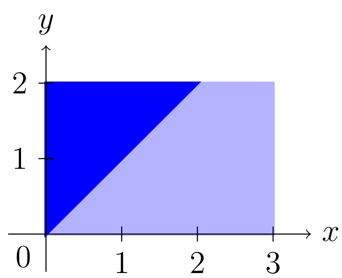

Figure 6: Setting up the integral for $P(Y > X)$.

Method #1: We can integrate with respect to y as the outer integral and with respect to x as the inner integral.

The desired probability is

$$\begin{aligned}
\int_0^2 \int_0^y \frac{1}{9}(3-x)(2-y)\,dx\,dy &= \int_0^2 \frac{1}{9}(3x - x^2/2)(2-y)\Big|_{x=0}^{y}\,dy \\
&= \int_0^2 \frac{1}{9}(3y - y^2/2)(2-y)\,dy \\
&= \int_0^2 \frac{1}{9}(6y - 4y^2 + y^3/2)\,dy \\
&= \frac{1}{9}\left(3y^2 - \frac{4}{3}y^3 + y^4/8\right)\Big|_{y=0}^{2} \\
&= \frac{1}{9}\left(3(2)^2 - \frac{4}{3}(2)^3 + (2)^4/8\right) \\
&= (1/9)(12 - 32/3 + 2) \\
&= 10/27
\end{aligned}$$

Method #2: We can integrate with respect to x as the outer integral and with respect to y as the inner integral.

The desired probability is

$$\begin{aligned}
\int_0^2 \int_x^2 \frac{1}{9}(3-x)(2-y)\,dy\,dx &= \int_0^2 \frac{1}{9}(3-x)(2y - y^2/2)\Big|_{y=x}^{2}\,dx \\
&= \int_0^2 \frac{1}{9}(3-x)(2 - 2x + x^2/2)\,dx \\
&= \int_0^2 \frac{1}{9}\left(6 - 8x + \frac{7}{2}x^2 - x^3/2\right)\,dx \\
&= \frac{1}{9}\left(6x - 4x^2 + \frac{7}{6}x^3 - x^4/8\right)\Big|_{x=0}^{2} \\
&= \frac{1}{9}\left(6(2) - 3(2)^2 + \frac{1}{2}(2)^3 - (2)^2 + \frac{2}{3}(2)^3 - (2)^4/8\right) \\
&= 10/27
\end{aligned}$$

STAT/MA 41600
Practice Problems: October 20, 2014
Solutions by Mark Daniel Ward

1.

a. No, X and Y are not independent. They are dependent. This is perhaps easiest to see because they are not defined on rectangles. So, for instance, $P(X > 2 \text{ and } Y > 2) = 0$ but $P(X > 2)P(Y > 2) \neq 0$.

b. The density of $f_X(x)$, for $0 \le x \le 3$, is $f_X(x) = \int_0^{3-x} 2/9\,dy = \frac{2}{9}y\big|_{y=0}^{3-x} = \frac{2}{9}(3-x)$. So

$$f_X(x) = \begin{cases} \frac{2}{9}(3-x) & \text{if } 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

c. The density of $f_Y(y)$, for $0 \le y \le 3$, is $f_Y(y) = \int_0^{3-y} 2/9\,dx = \frac{2}{9}x\big|_{x=0}^{3-y} = \frac{2}{9}(3-y)$. So

$$f_Y(y) = \begin{cases} \frac{2}{9}(3-y) & \text{if } 0 \le y \le 3 \\ 0 & \text{otherwise} \end{cases}$$

2.

a. No, X and Y are not independent. They are dependent. This is again perhaps easiest to see because they are not defined on rectangles. So, for instance, $P(X > 1 \text{ and } Y > 11) = 0$ but $P(X > 1)P(Y > 11) \neq 0$.

b. The density of $f_X(x)$, for $0 \le x \le 2$, is $f_X(x) = \int_0^{12-3x} 1/18\,dy = \frac{1}{18}y\big|_{y=0}^{12-3x} = \frac{1}{18}(12-3x) = \frac{1}{6}(4-x)$. So

$$f_X(x) = \begin{cases} \frac{1}{6}(4-x) & \text{if } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

c. The density of $f_Y(y)$, for $0 \le y \le 6$, is $f_Y(y) = \int_0^2 1/18\,dx = 1/9$.

The density of $f_Y(y)$, for $6 \le y \le 12$, is $f_Y(y) = \int_0^{(12-y)/3} 1/18\,dx = \frac{1}{18}x\big|_{x=0}^{(12-y)/3} = \frac{1}{18}(12-y)/3 = (12-y)/54$. So

$$f_Y(y) = \begin{cases} \frac{1}{9} & \text{if } 0 \le y \le 6 \\ \frac{12-y}{54} & \text{if } 6 \le y \le 12 \\ 0 & \text{otherwise} \end{cases}$$

3.

a. Yes, X and Y are independent, because their joint density $f_{X,Y}(x,y)$ can be factored into the x stuff times the y stuff, e.g., we can right $14e^{-2x-7y} = 14e^{-2x}e^{-7y}$.

b. The density of $f_X(x)$, for $x > 0$, is $f_X(x) = \int_0^\infty 14e^{-2x-7y}\,dy = -2e^{-2x-7y}\big|_{y=0}^{\infty} = 2e^{-2x}$. So

$$f_X(x) = \begin{cases} 2e^{-2x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

c. The density of $f_Y(y)$, for $y > 0$, is $f_Y(y) = \int_0^\infty 14e^{-2x-7y}\,dx = -7e^{-2x-7y}\big|_{x=0}^\infty = 7e^{-7y}$. So

$$f_Y(y) = \begin{cases} 7e^{-7y} & \text{if } y > 0 \\ 0 & \text{otherwise} \end{cases}$$

4.

a. Yes, X and Y are independent, because their joint density $f_{X,Y}(x,y)$ can be factored into the x stuff times the y stuff, e.g., we can right $1/16 = (1/4)(1/4)$, and these are the densities of X and Y, as we will see below.

b. The density of $f_X(x)$, for $-2 \le x \le 2$, is $f_X(x) = \int_{-2}^{2} 1/16\,dy = 1/4$. So

$$f_X(x) = \begin{cases} 1/4 & \text{if } -2 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

c. The density of $f_Y(y)$, for $-2 \le y \le 2$, is $f_Y(y) = \int_{-2}^{2} 1/16\,dx = 1/4$. So

$$f_Y(y) = \begin{cases} 1/4 & \text{if } -2 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

5.

a. Yes, X and Y are independent, because their joint density $f_{X,Y}(x,y) = \frac{1}{9}(3-x)(2-y)$ is already factored into the x stuff times the y stuff.

b. The density of $f_X(x)$, for $0 \le x \le 3$, is $f_X(x) = \int_0^2 \frac{1}{9}(3-x)(2-y)\,dy = \frac{1}{9}(3-x)\left(2y - \frac{y^2}{2}\right)\Big|_{y=0}^{2} = \frac{1}{9}(3-x)\left(2(2) - \frac{2^2}{2}\right) = \frac{2}{9}(3-x)$. So

$$f_X(x) = \begin{cases} \frac{2}{9}(3-x) & \text{if } 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

c. The density of $f_Y(y)$, for $0 \le y \le 2$, is $f_Y(y) = \int_0^3 \frac{1}{9}(3-x)(2-y)\,dx = \frac{1}{9}\left(3x - \frac{x^2}{2}\right)(2-y)\Big|_{x=0}^{3} = \frac{1}{9}\left(3(3) - \frac{3^2}{2}\right)(2-y) = \frac{1}{2}(2-y)$. So

$$f_Y(y) = \begin{cases} \frac{1}{2}(2-y) & \text{if } 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

PurdueX: 416.2x
Probability: Distribution Models & Continuous Random Variables

Solution to the problem sets

Unit 8: Conditional Distributions and Expected Values

STAT/MA 41600
Practice Problems: October 22, 2014
Solutions by Mark Daniel Ward

1.

a. The density of Y, for $0 \le y \le 3$, is $f_Y(y) = \frac{2}{9}(3-y)$, as we saw in the previous problem set. The joint density is 2/9. Thus, the conditional density of X given Y is

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{2/9}{\frac{2}{9}(3-y)} = \frac{1}{3-y}, \qquad \text{for } 0 \le x \le 3-y,$$

and $f_{X|Y}(x \mid y) = 0$ otherwise.

b. The conditional probability is $P(X \le 1 \mid Y = 1) = \int_0^1 f_{X|Y}(x \mid 1)\,dx = \int_0^1 \frac{1}{3-1}\,dx = 1/2$.

c. Using Bayes' Theorem, we have $P(X \le 1 \mid Y \le 1) = \frac{P(X \le 1 \text{ and } Y \le 1)}{P(Y \le 1)} = \frac{1/(9/2)}{(5/2)/(9/2)} = 2/5$.

2.

a. The density of Y, for $0 \le y \le 6$, is $f_Y(y) = 1/9$, as we saw in the previous problem set. The joint density is 1/18. Thus, the conditional density of X given Y is

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{1/18}{1/9} = 1/2, \qquad \text{for } 0 \le x \le 2,$$

and $f_{X|Y}(x \mid y) = 0$ otherwise.

b. The density of Y, for $6 \le y \le 12$, is $f_Y(y) = (12-y)/54$, as we saw in the previous problem set. The joint density is 1/18. Thus, the conditional density of X given Y is

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{1/18}{(12-y)/54} = 3/(12-y), \qquad \text{for } 0 \le x \le (12-y)/3,$$

and $f_{X|Y}(x \mid y) = 0$ otherwise.

c. Using Bayes' Theorem, $P(X \le 1 \mid 3 \le Y \le 9) = \frac{P(X \le 1 \text{ and } 3 \le Y \le 9)}{P(3 \le Y \le 9)} = \frac{6/18}{10.5/18} = 4/7$.

3.

a. Since X and Y are independent, then $f_{X|Y}(x \mid y) = f_X(x)$. Thus $f_{X|Y}(x \mid y) = 2e^{-2x}$ for $x > 0$, and $f_{X|Y}(x \mid y) = 0$ otherwise.

b. Since X and Y are independent, then $P(X \ge 1 \mid Y = 3) = P(X \ge 1) = \int_1^\infty 2e^{-2x}\,dx = -e^{-2x}\big|_{x=1}^\infty = e^{-2} = 0.1353$.

c. Since X and Y are independent, then then $f_{Y|X}(y \mid x) = f_Y(y)$. Thus $f_{Y|X}(y \mid x) = 7e^{-7y}$ for $y > 0$, and $f_{Y|X}(y \mid x) = 0$ otherwise. So $P(Y \le 1/5 \mid X = 2.7) = P(Y \le 1/5) = \int_0^{1/5} 7e^{-7y}\,dx = -e^{-7y}\big|_{y=0}^{1/5} = 1 - e^{-7/5} = 0.7534$.

4.

a. The density of $f_Y(y)$, for $y > 0$, is $f_Y(y) = \int_y^\infty 18e^{-2x-7y}\,dx = -9e^{-2x-7y}\big|_{x=y}^\infty = 9e^{-9y}$. So

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{18e^{-2x-7y}}{9e^{-9y}} = 2e^{-2x+2y} \qquad \text{if } x > y,$$

and $f_{X|Y}(x \mid y) = 0$ otherwise.

b. The density of $f_X(x)$, for $x > 0$, is $f_X(x) = \int_0^x 18e^{-2x-7y}\,dy = \frac{18}{7}\left(e^{-2x} - e^{-9x}\right)$. So

$$f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{18e^{-2x-7y}}{\frac{18}{7}\left(e^{-2x} - e^{-9x}\right)} = \frac{7e^{-7y}}{\left(1 - e^{-7x}\right)} \qquad \text{if } 0 < y < x,$$

and $f_{Y|X}(y \mid x) = 0$ otherwise.

5.

a. Since X and Y are independent, then $f_{X|Y}(x \mid y) = f_X(x)$. Thus $f_{X|Y}(x \mid y) = \frac{2}{9}(3-x)$ for $0 \le x \le 3$, and $f_{X|Y}(x \mid y) = 0$ otherwise.

b. Since X and Y are independent, $P(X \le 2 \mid Y = 3/2) = P(X \le 2) = \int_0^2 \frac{2}{9}(3-x)\,dx = \frac{2}{9}\left(3x - \frac{x^2}{2}\right)\Big|_{x=0}^{2} = 8/9$.

c. Since X and Y are independent, $P(Y \ge 1 \mid X = 5/4) = P(Y \ge 1) = \int_1^2 \frac{1}{2}(2-y)\,dy = \frac{1}{2}\left(2y - \frac{y^2}{2}\right)\Big|_{y=1}^{2} = 1/4$.

STAT/MA 41600
Practice Problems: October 24, 2014
Solutions by Mark Daniel Ward

1. As we saw earlier, the density of X is

$$f_X(x) = \begin{cases} \frac{2}{9}(3-x) & \text{if } 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

Thus $\mathbb{E}(X) = \int_0^3 \frac{2}{9}(3-x)x\,dx = \frac{2}{9}\int_0^3 (3x - x^2)\,dx = \frac{2}{9}\left(\frac{3x^2}{2} - \frac{x^3}{3}\right)\Big|_{x=0}^3 = \frac{2}{9}\left(\frac{27}{2} - \frac{27}{3}\right) = 1.$

2.

a. As we saw earlier, the density of X is

$$f_X(x) = \begin{cases} \frac{1}{6}(4-x) & \text{if } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Thus $\mathbb{E}(X) = \int_0^2 \frac{1}{6}(4-x)x\,dx = \frac{1}{6}\int_0^2 (4x - x^2)\,dx = \frac{1}{6}\left(\frac{4x^2}{2} - \frac{x^3}{3}\right)\Big|_{x=0}^2 = \frac{1}{6}\left(8 - \frac{8}{3}\right) = 8/9.$

b. As we saw earlier, the density of Y is

$$f_Y(y) = \begin{cases} \frac{1}{9} & \text{if } 0 \le y \le 6 \\ \frac{12-y}{54} & \text{if } 6 \le y \le 12 \\ 0 & \text{otherwise} \end{cases}$$

Thus $\mathbb{E}(Y) = \int_0^6 \frac{1}{9}y\,dy + \int_6^{12} \frac{12-y}{54}y\,dy = \frac{1}{9}\frac{y^2}{2}\Big|_{y=0}^6 + \int_6^{12} \frac{12y-y^2}{54}\,dy = \frac{1}{9}(18) + \frac{1}{54}\left(6y^2 - \frac{y^3}{3}\right)\Big|_{y=6}^{12} = 2 + \frac{1}{54}\left((864-576) - (216-72)\right) = 14/3.$

3.

a. As we saw earlier, the density of X is

$$f_X(x) = \begin{cases} 2e^{-2x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus, using $u = x$ and $dv = 2e^{-2x}$ in integration by parts, we have $du = dx$ and $v = -e^{-2x}$, so we get $\mathbb{E}(X) = \int_0^\infty 2e^{-2x}x\,dx = -xe^{-2x}\big|_{x=0}^\infty - \int_0^\infty -e^{-2x}\,dx = \frac{e^{-2x}}{-2}\Big|_{x=0}^\infty = \frac{1}{2}.$

b. As we saw earlier, the density of Y is

$$f_Y(y) = \begin{cases} 7e^{-7y} & \text{if } y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus, using $u = y$ and $dv = 7e^{-7y}$ in integration by parts, we have $du = dy$ and $v = -e^{-7y}$, so we get $\mathbb{E}(Y) = \int_0^\infty 7e^{-7y}y\,dy = -ye^{-7y}\big|_{y=0}^\infty - \int_0^\infty -e^{-7y}\,dy = \frac{e^{-7y}}{-7}\Big|_{y=0}^\infty = \frac{1}{7}.$

4.

a. As we saw earlier, the density of X is

$$f_X(x) = \begin{cases} \frac{18}{7}\left(e^{-2x} - e^{-9x}\right) & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus, $\mathbb{E}(X) = \int_0^\infty \frac{18}{7}\left(e^{-2x} - e^{-9x}\right) x\,dx = \frac{18}{7}\left(\int_0^\infty e^{-2x}x\,dx - \int_0^\infty e^{-9x}x\,dx\right)$. Using $u = x$ and $dv = e^{-2x}$ in integration by parts, we have $du = dx$ and $v = -e^{-2x}/2$, so $\int_0^\infty e^{-2x}x\,dx = -xe^{-2x}/2\big|_{x=0}^\infty - \int_0^\infty -e^{-2x}/2\,dx = \frac{e^{-2x}}{-4}\Big|_{x=0}^\infty = 1/4$. Similarly, using $u = x$ and $dv = e^{-9x}$ in integration by parts, we have $du = dx$ and $v = -e^{-9x}/9$, so $\int_0^\infty e^{-9x}x\,dx = -xe^{-9x}/9\big|_{x=0}^\infty - \int_0^\infty -e^{-9x}/9\,dx = \frac{e^{-9x}}{-81}\Big|_{x=0}^\infty = 1/81$. Thus $\mathbb{E}(X) = \frac{18}{7}\left(\frac{1}{4} - \frac{1}{81}\right) = 11/18$.

b. As we saw earlier, the density of Y is

$$f_Y(y) = \begin{cases} 9e^{-9y} & \text{if } y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus, using $u = y$ and $dv = 9e^{-9y}$ in integration by parts, we have $du = dy$ and $v = -e^{-9y}$, so we get $\mathbb{E}(Y) = \int_0^\infty 9e^{-9y}y\,dy = -ye^{-9y}\big|_{y=0}^\infty - \int_0^\infty -e^{-9y}\,dy = \frac{e^{-9y}}{-9}\Big|_{y=0}^\infty = \frac{1}{9}$.

5.

a. As we saw earlier, the density of X is

$$f_X(x) = \begin{cases} \frac{2}{9}(3 - x) & \text{if } 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

So, exactly as in Question 1, we have:

$\mathbb{E}(X) = \int_0^3 \frac{2}{9}(3 - x)x\,dx = \frac{2}{9}\int_0^3 (3x - x^2)\,dx = \frac{2}{9}\left(\frac{3x^2}{2} - \frac{x^3}{3}\right)\Big|_{x=0}^3 = \frac{2}{9}\left(\frac{27}{2} - 9\right) = 1.$

b. As we saw earlier, the density of X is

$$f_Y(y) = \begin{cases} \frac{1}{2}(2 - y) & \text{if } 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Thus $\mathbb{E}(Y) = \int_0^2 \frac{1}{2}(2 - y)y\,dy = \frac{1}{2}\int_0^2 (2y - y^2)\,dy = \frac{1}{2}\left(y^2 - \frac{y^3}{3}\right)\Big|_{y=0}^2 = \frac{1}{2}\left(4 - \frac{8}{3}\right) = 2/3.$

STAT/MA 41600
Practice Problems: October 27, 2014
Solutions by Mark Daniel Ward

1.

a. *Method #1:* Since we saw $\mathbb{E}(X) = 1$ and $\mathbb{E}(Y) = 1$ in the previous problem set, then $\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y) = 1 + 1 = 2$.

Method #2: We have $\mathbb{E}(X+Y) = \int_0^3 \int_0^{3-x} (x+y)(\frac{2}{9})\,dy\,dx = \int_0^3 \left(xy + \frac{y^2}{2}\right)\left(\frac{2}{9}\right)\Big|_{y=0}^{3-x} dx = \int_0^3 (1 - \frac{x^2}{9})\,dx = (x - \frac{x^2}{9})\Big|_{x=0}^{3} = (3 - \frac{3^2}{9}) = 2$.

b. *Method #1:* Since we know from the previous problem set that $f_X(x) = \frac{2}{9}(3-x)$ for $0 \le x \le 3$, then we can integrate $\mathbb{E}(X^2) = \int_0^3 x^2(\frac{2}{9})(3-x)\,dx = \int_0^3 (\frac{2}{9})(3x^2 - x^3)\,dx = (\frac{2}{9})(x^3 - \frac{x^4}{4})\Big|_{x=0}^{3} = (\frac{2}{9})(27 - \frac{81}{4}) = 3/2$. So $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 3/2 - 1^2 = 1/2$.

Method #2: We can integrate $\mathbb{E}(X^2) = \int_0^3 \int_0^{3-x} (x^2)(\frac{2}{9})\,dy\,dx = \int_0^3 (3x^2 - x^3)(\frac{2}{9})\,dx = \frac{2}{9}(x^3 - \frac{x^4}{4})\Big|_{x=0}^{3} = \frac{2}{9}(27 - \frac{81}{4}) = 3/2$. So $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 3/2 - 1^2 = 1/2$.

2.

a. *Method #1:* Since we know from the previous problem set that $f_X(x) = \frac{1}{6}(4-x)$ for $0 \le x \le 2$, then we can integrate $\mathbb{E}(X^2) = \int_0^2 x^2(\frac{1}{6})(4-x)\,dx = \int_0^2 (\frac{1}{6})(4x^2 - x^3)\,dx = (\frac{1}{6})(\frac{4x^3}{3} - \frac{x^4}{4})\Big|_{x=0}^{2} = (\frac{1}{6})(\frac{32}{3} - 4) = 10/9$. We saw $\mathbb{E}(X) = 8/9$ in the previous problem set. So $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 10/9 - (8/9)^2 = 26/81$.

Method #2: We have $\mathbb{E}(X^2) = \int_0^2 \int_0^{12-3x} (x^2)(\frac{1}{18})\,dy\,dx = \int_0^2 (\frac{1}{18})(12x^2 - 3x^3)\,dx = (\frac{1}{18})(4x^3 - \frac{3x^4}{4})\Big|_{x=0}^{2} = 10/9$. So $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 10/9 - (8/9)^2 = 26/81$.

b. *Method #1:* Since we know from the previous problem set that $f_Y(y) = 1/6$ for $0 \le y \le 6$ and $f_Y(y) = \frac{12-y}{54}$ for $6 \le x \le 12$, then we can integrate $\mathbb{E}(Y^2) = \int_0^6 y^2(\frac{1}{9})\,dy + \int_6^{12} y^2(\frac{12-y}{54})\,dy = \frac{6^3}{3}(\frac{1}{9}) + (\frac{12y^3/3 - y^4/4}{54})\Big|_{y=6}^{12} = 30$. We saw $\mathbb{E}(Y) = 14/3$ in the previous problem set. So $\text{Var}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = 30 - (14/3)^2 = 74/9$.

Method #2: We have $\mathbb{E}(Y^2) = \int_0^2 \int_0^{12-3x} (y^2)(\frac{1}{18})\,dy\,dx = \int_0^2 (\frac{1}{18})\frac{(12-3x)^3}{3}\,dx$. Using $u = 12 - 3x$, $du = -3dx$, we get $\mathbb{E}(Y^2) = \int_6^{12} (\frac{1}{18})\frac{u^3}{9}\,du = \frac{1}{18}(\frac{12^4}{36} - \frac{6^4}{36}) = 30$. So $\text{Var}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = 30 - (14/3)^2 = 74/9$.

3. Since X and Y are independent, then $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$. We already saw in the previous problem set that $f_X(x) = 2e^{-2x}$ for $x > 0$, and $f_X(x) = 0$ otherwise; also $\mathbb{E}(X) = 1/2$. We already saw $f_Y(x) = 7e^{-7y}$ for $y > 0$, and $f_Y(y) = 0$ otherwise; also $\mathbb{E}(Y) = 1/7$.

Now we compute $\mathbb{E}(X^2) = \int_0^\infty x^2 2e^{-2x}\,dx$, and we use $u = x^2$ and $dv = 2e^{-2x}\,dx$, so $du = 2x\,dx$ and $v = -e^{-2x}$, to get $\mathbb{E}(X^2) = -x^2e^{-2x}|_{x=0}^\infty - \int_0^\infty -2xe^{-2x}\,dx = \int_0^\infty x\,2e^{-2x}\,dx$. We can either integrate a second time, by parts, or just recognize that the integral here is equal to $\mathbb{E}(X)$, which we already calculated in the previous problem set, question #3. So altogether we have $\mathbb{E}(X^2) = \mathbb{E}(X) = 1/2$. Thus $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 1/2 - (1/2)^2 = 1/4$.

Similarly $\mathbb{E}(Y^2) = \int_0^\infty y^2 7e^{-7y}\,dy$, and we use $u = y^2$ and $dv = 7e^{-7y}\,dy$, so $du = 2y\,dy$ and $v = -e^{-7y}$, to get $\mathbb{E}(Y^2) = -y^2e^{-7y}|_{y=0}^\infty - \int_0^\infty -2ye^{-7y}\,dy = \frac{2}{7}\int_0^\infty y\,7e^{-7y}\,dy$. We can either integrate a second time, by parts, or just recognize that the integral here is equal to $\mathbb{E}(Y)$, which we already calculated in the previous problem set, #3. So altogether $\mathbb{E}(Y^2) = \frac{2}{7}\mathbb{E}(Y) = (2/7)(1/7) = 2/49$. Thus $\mathrm{Var}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = 2/49 - (1/7)^2 = 1/49$.

4. *Method #1:* We know $\mathbb{E}(Y) = 1/9$ from the previous problem set, and $f_Y(y) = 9e^{-9y}$ for $y > 0$, and $f_Y(y) = 0$ otherwise. Also $\mathbb{E}(Y^2) = \int_0^\infty (y^2)(9e^{-9y})\,dy$, and we use $u = y^2$ and $dv = 9e^{-9y}\,dy$, so $du = 2y\,dy$ and $v = -e^{-9y}$, to get $\mathbb{E}(Y^2) = -y^2e^{-9y}|_{y=0}^\infty - \int_0^\infty -2ye^{-9y}\,dy = \frac{2}{9}\int_0^\infty y\,9e^{-9y}\,dy$. We can either integrate a second time, by parts, or just recognize that the integral here is equal to $\mathbb{E}(Y)$, which is 1/9, as in the previous problem set, #4. So altogether $\mathbb{E}(Y^2) = \frac{2}{9}\mathbb{E}(Y) = (2/9)(1/9) = 2/81$. Thus $\mathrm{Var}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = 2/81 - (1/9)^2 = 1/81$.

Method #2: We compute $\mathbb{E}(Y^2) = \int_0^\infty \int_y^\infty (y^2)(18e^{-2x-7y})\,dx\,dy = \int_0^\infty (y^2)(-9e^{-2x-7y})|_{x=y}^\infty\,dy = \int_0^\infty (y^2)(9e^{-9y})\,dy$, and then everything else proceeds as in Method #1 above, i.e., we get $\mathbb{E}(Y^2) = 2/81$ in the same way from Method #1, starting on the second line. We also have $\mathbb{E}(Y) = 1/9$, so $\mathrm{Var}(Y) = 2/81 - (1/9)^2 = 1/81$.

5. We have $\mathbb{E}(X^2 + Y^3) = \mathbb{E}(X^2) + \mathbb{E}(Y^3)$.

As in the previous problem set, $f_X(x) = \frac{2}{9}(3-x)$ for $0 \le x \le 3$ and $f_X(x) = 0$ otherwise. So $\mathbb{E}(X^2) = \int_0^3 (x^2)(\frac{2}{9})(3-x)\,dx = \int_0^3 \frac{2}{9}(3x^2 - x^3)\,dx = \frac{2}{9}(x^3 - x^4/4)\big|_{x=0}^3 = 3/2$.

As in the previous problem set, $f_Y(y) = \frac{1}{2}(2-y)$ for $0 \le y \le 2$ and $f_Y(y) = 0$ otherwise. So $\mathbb{E}(Y^3) = \int_0^2 (y^3)(\frac{1}{2})(2-y)\,dy = \int_0^2 \frac{1}{2}(2y^3 - y^4)\,dy = \frac{1}{2}(2y^4/4 - y^5/5)\big|_{y=0}^2 = 4/5$.

Thus $\mathbb{E}(X^2 + Y^3) = \mathbb{E}(X^2) + \mathbb{E}(Y^3) = 3/2 + 4/5 = 23/10 = 2.3$.

PurdueX: 416.2x
Probability: Distribution Models & Continuous Random Variables

Solution to the problem sets

Unit 9: Models of Continuous Random Variables

STAT/MA 41600
Practice Problems: October 29, 2014
Solutions by Mark Daniel Ward

1.

a. The probability is $P(X \le 4.5) = F_X(4.5) = \frac{4.5-2}{4} = 0.625$.

b. *Method #1:* The probability is $P(3.09 \le X \le 4.39) = P(X \le 4.39) - P(X < 3.09) = F_X(4.39) - F_X(3.09) = \frac{4.39-2}{4} - \frac{3.09-2}{4} = 0.325$.

Method #2: The density is $f_X(x) = \frac{d}{dx}F_X(x) = 1/4$ for $2 \le x \le 6$ and $f_X(x) = 0$ otherwise. So $P(3.09 \le X \le 4.39) = \int_{3.09}^{4.39} 1/4\,dx = 1.3/4 = 0.325$.

Method #3: Since X has continuous uniform distribution, we can use the lengths of the line segments, to compute $P(3.09 \le X \le 4.39) = \frac{\text{length of }[3.09,4.39]}{\text{length of }[2,6]} = \frac{1.3}{4} = 0.325$.

c. The probability is $P(X \ge 3.7) = 1 - P(X < 3.7) = 1 - F_X(3.7) = 1 - \frac{3.7-2}{4} = 0.575$.

2.

a. *Method #1:* Using the CDF formula, the probability is $P(X > 12) = 1 - P(X \le 12) = 1 - F_X(12) = 1 - \frac{12-11.93}{12.02-11.93} = 0.222$.

Method #2: The density is $f_X(x) = \frac{1}{12.02-11.93} = \frac{1}{0.09}$ for $11.93 \le x \le 12.02$ and $f_X(x) = 0$ otherwise. So $P(X \ge 12) = \int_{12}^{12.02} \frac{1}{0.09}\,dx = \frac{0.02}{0.09} = 0.222$.

Method #3: Since X has continuous uniform distribution, we can use the lengths of the line segments, to compute $P(X \ge 12) = \frac{\text{length of }[12,12.02]}{\text{length of }[11.93,12.02]} = \frac{0.02}{0.09} = 0.222$.

b. Since the amount of soda is uniform on the interval $[11.93, 12.02]$, then the variance is $(12.02 - 11.93)^2/12 = 0.000675$, so the standard deviation is $\sqrt{0.000675} = 0.02598$ ounces.

3.

a. We write X as the quantity of gasoline, so that X is uniform on $[4.30, 4.50]$ and the cost of the purchase is $12X + 1.00$. So $\mathbb{E}(X) = (4.30 + 4.50)/2 = 4.40$, and thus the expected value of the cost of the purchase is $12(4.40) + 1.00 = 53.80$ dollars.

b. Using the notation from part (a), we have $\text{Var}(X) = (4.50 - 4.30)^2/12 = 0.003333$. Thus, the variance of the purchase cost is $\text{Var}(12X+1.00) = 12^2\,\text{Var}(X) = (144)(0.003333) = 0.48$.

4. *Method #1:* The three random variables X, Y, Z are independent and identically distributed, so any of the three of them is equally-likely to be the middle value. Thus Y is the middle value with probability 1/3.

Method #2: Each of the random variables has density 1/10, so the joint density is $f_{X,Y,Z}(x, y, z) = 1/1000$. Thus, we can integrate

$$P(X < Y < Z) = \int_0^{10}\int_0^{z}\int_0^{y} \frac{1}{1000}\,dx\,dy\,dz = \int_0^{10}\int_0^{z} \frac{y}{1000}\,dy\,dz = \int_0^{10} \frac{z^2/2}{1000}\,dz = \frac{10^3/6}{1000} = 1/6,$$

and

$$P(Z < Y < X) = \int_0^{10}\int_0^{x}\int_0^{y} \frac{1}{1000}\,dz\,dy\,dx = \int_0^{10}\int_0^{x} \frac{y}{1000}\,dy\,dx = \int_0^{10} \frac{x^2/2}{1000}\,dx = \frac{10^3/6}{1000} = 1/6,$$

so we add the probabilities of these disjoint events: $P(X < Y < Z \text{ or } Z < Y < X) = P(X < Y < Z) + P(Z < Y < X) = 1/6 + 1/6 = 1/3$.

5. We use Figure 1 to guide the way to setup the integral. The joint density of X and Y, as we have seen in previous problem sets, is $f_{X,Y}(x,y) = 2/9$ for X, Y in the triangle, and $f_{X,Y}(x,y) = 0$ otherwise. So we have

$$\begin{aligned}
\mathbb{E}(\min(X,Y)) &= \int_0^{3/2}\int_0^{x} \frac{2}{9}y\,dy\,dx + \int_{3/2}^{3}\int_0^{3-x} \frac{2}{9}y\,dy\,dx + \int_0^{3/2}\int_0^{y} \frac{2}{9}x\,dx\,dy + \int_{3/2}^{3}\int_0^{3-y} \frac{2}{9}x\,dx\,dy \\
&= \int_0^{3/2} \frac{x^2}{9}\,dx + \int_{3/2}^{3} \frac{(3-x)^2}{9}\,dx + \int_0^{3/2} \frac{y^2}{9}\,dy + \int_{3/2}^{3} \frac{(3-y)^2}{9}\,dy \\
&= 1/8 + 1/8 + 1/8 + 1/8 \\
&= 1/2
\end{aligned}$$

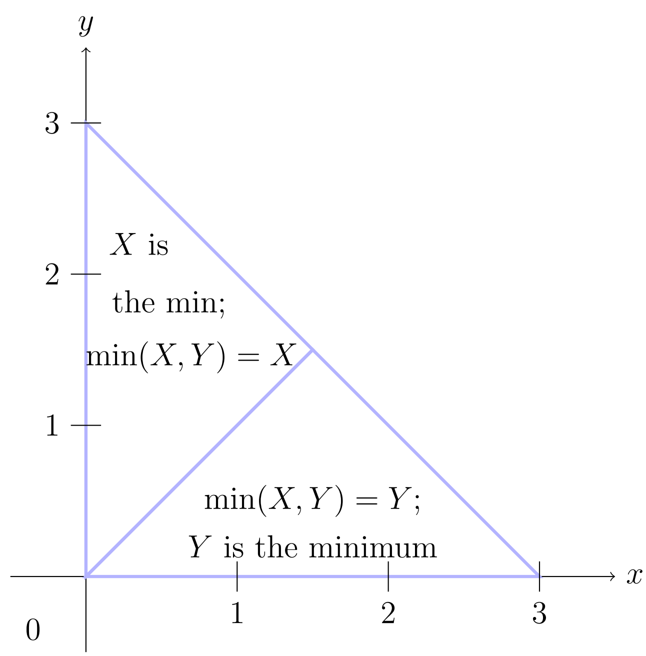

Figure 1: The regions where $\min(X,Y) = X$ versus where $\min(X,Y) = Y$.

STAT/MA 41600
Practice Problems: October 31, 2014
Solutions by Mark Daniel Ward

1.

a. The probability is $\int_{60}^{\infty} \frac{1}{30} e^{-x/30}\,dx = -e^{-x/30}\Big|_{x=60}^{\infty} = e^{-2} = .1353$.

b. Since the average is $1/\lambda = 30$ minutes, then the variance is $1/\lambda^2 = 900$, so the standard deviation is $\sqrt{1/\lambda^2} = 30$ minutes.

2. Let X be the time (in minutes) until the next dessert; let Y be the time (in minutes) until the next appetizer. The probability is $P(X < Y) = \int_0^{\infty} \int_x^{\infty} \frac{1}{3} e^{-x/3} \frac{1}{2} e^{-y/2}\,dy\,dx = \int_0^{\infty} -\frac{1}{3} e^{-x/3} e^{-y/2}\Big|_{y=x}^{\infty}\,dx = \int_0^{\infty} \frac{1}{3} e^{-5x/6}\,dx = -\frac{1/3}{5/6} e^{-5x/6}\Big|_{x=0}^{\infty} = 2/5$.

3. Using the CDF of X, we have $P(X \leq 20) = F_X(20) = 1 - e^{-20(1/20)} = 1 - e^{-1}$. Similarly $P(Y \leq 20) = 1 - e^{-1}$ and $P(Z \leq 20) = 1 - e^{-1}$. Since the X, Y, Z are independent, then $P(\max(X, Y, Z)) = P(X \leq 20)P(Y \leq 20)P(Z \leq 20) = (1 - e^{-1})^3 = 0.2526$.

4. The company expects to pay $\int_0^3 (0) \frac{1}{1.5} e^{-x/1.5}\,dx + \int_3^{\infty} (72)(100)(x - 3) \frac{1}{1.5} e^{-x/1.5}\,dx = (72)(100) \int_0^{\infty} x \frac{1}{1.5} e^{-(x+3)/1.5}\,dx = (72)(100) e^{-2} \int_0^{\infty} x \frac{1}{1.5} e^{-x/1.5}\,dx$. Notice $\int_0^{\infty} x \frac{1}{1.5} e^{-x/1.5}\,dx$ is the expected value of X, i.e., is 1.5. So the company expects to pay $(72)(100)(e^{-2})(1.5) = 1461.62$ dollars.

5. The probability is
$\int_0^{10} \int_x^{\infty} (\frac{1}{10})(\frac{1}{5}) e^{-y/5}\,dy\,dx = \int_0^{10} -\frac{1}{10} e^{-y/5}\Big|_{y=x}^{\infty}\,dx = \int_0^{10} \frac{1}{10} e^{-x/5}\,dx = \frac{1}{2} \int_0^{10} \frac{1}{5} e^{-x/5}\,dx$, but the last integral is just the CDF of an exponential with average of 5, evaluated at 10. So the overall probability is $\frac{1}{2}(1 - e^{-10/5}) = \frac{1}{2}(1 - e^{-2}) = 0.4323$.

STAT/MA 41600
Practice Problems: November 5, 2014
Solutions by Mark Daniel Ward

1. a. *Method #1:* Since Y is a Gamma random variable with $1/\lambda = 30$ and $r = 3$, then $\mathbb{E}(Y) = r/\lambda = 90$ minutes.

Method #2: We can just add the expected values: $\mathbb{E}(Y) = \mathbb{E}(X_1 + X_2 + X_3) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \mathbb{E}(X_3) = 30 + 30 + 30 = 90$.

b. *Method #1:* Since Y is a Gamma random variable with $1/\lambda = 30$ and $r = 3$, then $\text{Var}(Y) = r/\lambda^2 = 2700$, so $\sigma_Y = \sqrt{2700} = 51.96$ minutes.

Method #2: Since X_1, X_2, X_3 are independent, we can just add the variances: $\text{Var}(Y) = \text{Var}(X_1 + X_2 + X_3) = \text{Var}(X_1) + \text{Var}(X_2) + \text{Var}(X_3) = 900 + 900 + 900 = 2700$, so $\sigma_Y = \sqrt{2700} = 51.96$ minutes.

2. *Method #1:* We notice that X is Gamma with $1/\lambda = 3$ and $r = 2$, so the density of X is $f_X(x) = \frac{(1/3)^2}{\Gamma(2)} x^{2-1} e^{-x/3} = \frac{1}{9} x e^{-x/3}$ for $x > 0$, and $f_X(x) = 0$ otherwise.

Method #2: The CDF of X, for $a > 0$, is $P(X \leq a) = \int_0^a \int_0^{a-x} \frac{1}{3} e^{-x/3} \frac{1}{3} e^{-y/3}\,dy\,dx = \int_0^a \frac{1}{3} e^{-x/3} \left(1 - e^{-(a-x)/3}\right)\,dx = \int_0^a \left(\frac{1}{3} e^{-x/3} - \frac{1}{3} e^{-a/3}\right)\,dx = \left(-e^{-x/3} - \frac{1}{3} e^{-a/3} x\right)\Big|_{x=0}^a = 1 - e^{-a/3} - \frac{1}{3} e^{-a/3} a$. Thus, $F_X(x) = 1 - e^{-x/3} - \frac{1}{3} e^{-x/3} x$ for $x > 0$, and $F_X(x) = 0$ otherwise. Differentiating with respect to x, we get $f_X(x) = \frac{1}{3} e^{-x/3} + \frac{1}{9} e^{-x/3} x - \frac{1}{3} e^{-x/3} = \frac{1}{9} e^{-x/3} x$ for $x > 0$, and $f_X(x) = 0$ otherwise.

3. *Method #1:* Since X is a Gamma random variable with $1/\lambda = 20$ and $r = 3$, then $\text{Var}(X) = r/\lambda^2 = 1200$.

Method #2: Since the waiting times are independent, we can just add the variances: $\text{Var}(X) = 400 + 400 + 400 = 1200$.

4. a. *Method #1:* We can just compute, treating Y as a function of X. We have $\mathbb{E}(Y) = \int_0^5 (0) \frac{1}{3} e^{-x/3} dx + \int_5^\infty (x-5) \frac{1}{3} e^{-x/3} dx = \int_5^\infty (x-5) \frac{1}{3} e^{-x/3} dx = \int_0^\infty x \frac{1}{3} e^{-(x+5)/3} dx$. We can factor out $e^{-5/3}$, so $\mathbb{E}(Y) = e^{-5/3} \int_0^\infty x \frac{1}{3} e^{-x/3}\,dx$, but the integral is 3, so $\mathbb{E}(Y) = 3e^{-5/3} = 0.5666$.

Method #2: The probability that $X \leq 5$ is $1 - e^{-5/3}$, and in this case, $Y = 0$. On the other hand, the probability that $X > 5$ is $e^{-5/3}$, and we know that, given $X > 5$, it follows that the conditional distribution of $X - 5$ is exponential with expected value 3. Thus $Y = X - 5$ has expected value 3 in this case. So the expected value of Y is $\mathbb{E}(Y) = (0)(1 - e^{-5/3}) + (3)(e^{-5/3}) = 3e^{-5/3} = 0.5666$.

b. *Method #1:* We can just compute, treating Y^2 as a function of X. We have $\mathbb{E}(Y^2) = \int_0^5 (0)^2 \frac{1}{3} e^{-x/3}\,dx + \int_5^\infty (x-5)^2 \frac{1}{3} e^{-x/3}\,dx = \int_5^\infty (x-5)^2 \frac{1}{3} e^{-x/3}\,dx = \int_0^\infty x^2 \frac{1}{3} e^{-(x+5)/3}\,dx$. We can factor out $e^{-5/3}$, so $\mathbb{E}(Y^2) = e^{-5/3} \int_0^\infty x^2 \frac{1}{3} e^{-x/3}\,dx$, but the integral is $2/\lambda^2 = 2(3^2) = 18$ (i.e., the second moment of an exponential, as on page 459), so $\mathbb{E}(Y^2) = 18e^{-5/3}$. So the variance of Y is $\text{Var}(Y) = 18e^{-5/3} - (3e^{-5/3})^2 = 18e^{-5/3} - 9e^{-10/3} = 3.0787$.

Method #2: The probability that $X \leq 5$ is $1 - e^{-5/3}$, and in this case, $Y^2 = 0$. On the other hand, the probability that $X > 5$ is $e^{-5/3}$, and we know that, given $X > 5$, it follows that the conditional distribution of $X - 5$ is exponential with expected value 3. Thus $Y = X - 5$ has $\mathbb{E}(Y^2) = 2/\lambda^2 = 2(3^2) = 18$ in this case. So the expected value of Y^2 is $\mathbb{E}(Y^2) = (0)(1 - e^{-5/3}) + (18)(e^{-5/3}) = 18e^{-5/3}$, and the variance of Y is $\text{Var}(Y) = 18e^{-5/3} - (3e^{-5/3})^2 = 18e^{-5/3} - 9e^{-10/3} = 3.0787$.

5. For $a > 0$, we have $P(X > a) = (e^{-a/5})^3 = e^{-(3/5)a}$. Thus $F_X(x) = 1 - e^{-(3/5)x}$ for $x > 0$ and $F_X(x) = 0$ otherwise. So X is exponential with $\mathbb{E}(X) = 5/3$.

PurdueX: 416.2x
Probability: Distribution Models & Continuous Random Variables

Solution to the problem sets

Unit 10: Normal Distribution and Central Limit Theorem (CLT)

STAT/MA 41600
Practice Problems: November 10, 2014
Solutions by Mark Daniel Ward

1. a. We compute $P(X \le 10) = P\big(\frac{X-4.2}{\sqrt{50.41}} \le \frac{10-4.2}{\sqrt{50.41}}\big) = P(Z \le 0.82) = 0.7939$.

b. We compute $P(X \le 0) = P\big(\frac{X-4.2}{\sqrt{50.41}} \le \frac{0-4.2}{\sqrt{50.41}}\big) = P(Z \le -0.59) = P(0.59 \le Z) = 1 - P(Z \le 0.59) = 1 - 0.7224 = 0.2776$.

c. Combining the work above, we have $P(0 \le X \le 10) = P(X \le 10) - P(X \le 0) = 0.7939 - 0.2776 = 0.5163$.

2. We compute $P(70 \le X) = P\big(\frac{70-72.5}{6.9} \le \frac{X-72.5}{6.9}\big) = P(-0.36 \le Z) = P(Z \le 0.36) = 0.6406$.

3. We compute $0.3898 = P(a \le Z \le .54) = P(Z \le .54) - P(Z \le a) = .7054 - P(Z \le a)$. Thus $P(Z \le a) = .7054 - 0.3898 = 0.3156$. [Note, in particular, that now we can see a will be negative.] Equivalently, we have $P(-a \le Z) = 0.3156$, so $P(Z \le -a) = 1 - 0.3156 = .6844$. So from the normal chart, we have $-a = 0.48$, so $a = -0.48$.

4. a. We compute $P(66 \le X) = P\big(\frac{66-64}{12.8} \le \frac{X-64}{12.8}\big) = P(0.16 \le Z) = 1 - P(Z \le 0.16) = 1 - 0.5636 = 0.4364$.

b. Let $X_1, \ldots, X_{10}$ be indicator random variables corresponding to the first, …, tenth person, so that $X_j = 1$ if the jth person has height 66 inches or taller, or $X_j = 0$ otherwise. Then $\mathbb{E}(X_1 + \cdots + X_{10}) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_{10}) = 0.4364 + \cdots + 0.4364 = 4.364$.

5. *Method #1:* We compute $0.1492 = P(X \le x) = P\big(\frac{X-22}{\sqrt{8}} \le \frac{x-22}{\sqrt{8}}\big) = P\big(Z \le \frac{x-22}{\sqrt{8}}\big)$. Taking complements on both sides yields $1 - 0.1492 = 1 - P\big(Z \le \frac{x-22}{\sqrt{8}}\big) = P\big(\frac{x-22}{\sqrt{8}} \le Z\big)$. Simplifying (and switching directions on the right-hand-side) yields $0.8508 = P\big(Z \le -\frac{x-22}{\sqrt{8}}\big)$. So $-\frac{x-22}{\sqrt{8}} = 1.04$, and thus $x = (\sqrt{8})(-1.04) + 22 = 19.06$.

Method #2: We start with $0.1492 = P(Z \le z)$, which is not on the table, so taking complements gives $1 - 0.1492 = 1 - P(Z \le z) = P(z \le Z)$, so $0.8508 = P(Z \le -z)$. Thus $-z = 1.04$, so $z = -1.04$. Now that we have the value of z we need, we can return to the original statement, to get: $0.1492 = P(Z \le -1.04) = P(\mu_X + \sigma_X Z \le \mu_X + \sigma_X(-1.04)) = P(X \le 22 - (\sqrt{8})(1.04)) = P(X \le 19.06)$. So the desired quantity is $x = 19.06$.

STAT/MA 41600
Practice Problems: November 12, 2014
Solutions by Mark Daniel Ward

We always use Z to denote a standard normal random variable in these answers.

1a. We have $\mathbb{E}(Y) = \mathbb{E}(\frac{X_1+X_2+X_3+X_4+X_5}{5}) = \frac{1}{5}(\mathbb{E}(X_1)+\cdots+\mathbb{E}(X_5)) = \frac{1}{5}(8.2+\cdots+8.2) = 8.2$. The X_j's are independent, so $\mathrm{Var}(Y) = \mathrm{Var}(\frac{X_1+X_2+X_3+X_4+X_5}{5}) = \frac{1}{25}(\mathrm{Var}(X_1)+\cdots+\mathrm{Var}(X_5)) = \frac{1}{25}(32.49+\cdots+32.49) = \frac{32.49}{5} = 6.498$.

1b. We have $\mathbb{E}(Y) = \mathbb{E}(\frac{X_1+X_2+\cdots+X_n}{n}) = \frac{1}{n}(\mathbb{E}(X_1)+\cdots+\mathbb{E}(X_n)) = \frac{1}{n}(\mu+\cdots+\mu) = \mu$. The X_j's are independent, so $\mathrm{Var}(Y) = \mathrm{Var}(\frac{X_1+X_2+\cdots+X_n}{n}) = \frac{1}{n^2}(\mathrm{Var}(X_1)+\cdots+\mathrm{Var}(X_n)) = \frac{1}{n^2}(\sigma^2+\cdots+\sigma^2) = \frac{\sigma^2}{n}$.

2. Let Y_1, Y_2, Y_3 be the amounts in the three people's accounts. So $X = Y_1+Y_2+Y_3$. So X is the sum of independent normals, and thus X is normal too, with $\mathbb{E}(X) = \mathbb{E}(Y_1+Y_2+Y_3) = \mathbb{E}(Y_1)+\mathbb{E}(Y_2)+\mathbb{E}(Y_3) = 1325+1325+1325 = 3975$, and $\mathrm{Var}\,X = \mathrm{Var}(Y_1+Y_2+Y_3) = \mathrm{Var}(Y_1)+\mathrm{Var}(Y_2)+\mathrm{Var}(Y_3) = 25^2+25^2+25^2 = 1875$, so $\sigma_X = 43.30$. Thus $P(X > 4000) = P(\frac{X-3975}{43.30} > \frac{4000-3975}{43.30}) = P(Z > .58) = 1 - P(Z \le .58) = 1 - .7190 = .2810$.

3. Let X_1, X_2, X_3, X_4 be the lengths of time for the four people's haircuts. So $Y = X_1 + X_2 + X_3 + X_4$ is the total length of time. So Y is the sum of independent normals, and thus Y is normal too, with $\mathbb{E}(Y) = \mathbb{E}(X_1+X_2+X_3+X_4) = \mathbb{E}(X_1)+\mathbb{E}(X_2)+\mathbb{E}(X_3)+\mathbb{E}(X_4) = 23.8+23.8+23.8+23.8 = 95.2$, and $\mathrm{Var}\,Y = \mathrm{Var}(X_1+X_2+X_3+X_4) = \mathrm{Var}(X_1)+\mathrm{Var}(X_2)+\mathrm{Var}(X_3)+\mathrm{Var}(X_4) = 5^2+5^2+5^2+5^2 = 100$, so $\sigma_Y = 10$. Thus $P(Y \le 90) = P(\frac{Y-95.2}{10} \le \frac{90-95.2}{10}) = P(Z \le -.52) = P(Z \ge .52) = 1 - P(Z < .52) = 1 - .6985 = .3015$.

4. As in problem 1b above, $\mathbb{E}(Y) = 64$, and $\mathrm{Var}(Y) = \frac{12.8^2}{10} = 16.384$, so $P(Y > 60) = P(\frac{Y-64}{\sqrt{16.384}} > \frac{60-64}{\sqrt{16.384}}) = P(Z > -0.99) = P(Z < 0.99) = .8389$.

5. Let $Y = X_1 + \cdots + X_7$ be the total quantity of sugar, where X_j is the amount of sugar in the jth piece. So Y is the sum of independent normals, and thus Y is normal too, with $\mathbb{E}(Y) = \mathbb{E}(X_1+\cdots+X_7) = \mathbb{E}(X_1)+\cdots+\mathbb{E}(X_7) = 22+\cdots+22 = 154$, and $\mathrm{Var}\,Y = \mathrm{Var}(X_1+\cdots+X_7) = \mathrm{Var}(X_1)+\cdots+\mathrm{Var}(X_7) = 8+\cdots+8 = 56$, so $\sigma_Y = 7.48$. Thus $P(Y \ge 150) = P(\frac{Y-154}{7.48} \ge \frac{150-154}{7.48}) = P(Z \ge -.53) = P(Z \le .53) = .7019$.

STAT/MA 41600
Practice Problems: November 14, 2014
Solutions by Mark Daniel Ward

1. Let $X_1, \ldots, X_{30}$ denote the 30 waiting times. Then $\mathbb{E}(X_j) = 1/2$ and $\operatorname{Var} X_j = 1/4$, so $\mathbb{E}(X_1 + \cdots + X_{30}) = (30)(1/2) = 15$ and $\operatorname{Var}(X_1 + \cdots + X_{30}) = (30)(1/4) = 7.5$. So $P(X_1 + \cdots + X_{30} > 14) = P(\frac{X_1+\cdots+X_{30}-15}{\sqrt{7.5}} > \frac{14-15}{\sqrt{7.5}}) \approx P(Z > -.37) = P(Z < .37) = .6443$.

2. Let $X_1, \ldots, X_{30}$ be indicator random variables that denote whether the 30 students are happy with their items, i.e., $X_j = 1$ if the jth student is happy, or $X_j = 0$ otherwise. Then $\mathbb{E}(X_j) = .60$ and $\operatorname{Var} X_j = (.60)(.40) = .24$, so $\mathbb{E}(X_1 + \cdots + X_{30}) = (30)(.60) = 18$ and $\operatorname{Var}(X_1 + \cdots + X_{30}) = (30)(.24) = 7.2$. So, using continuity correction since the X_j's are integer-valued random variables, $P(X_1 + \cdots + X_{30} \geq 20) = P(X_1 + \cdots + X_{30} \geq 19.5) = P(\frac{X_1+\cdots+X_{30}-18}{\sqrt{7.2}} \geq \frac{19.5-18}{\sqrt{7.2}}) \approx P(Z \geq .56) = 1 - P(Z < .56) = 1 - .7123 = .2877$.

3. a. We compute $\mathbb{E}(X) = \int_0^{10} x\frac{(10-x)^3}{2500}\,dx = \int_0^{10}(10-u)\frac{u^3}{2500}\,du = 2$, and $\mathbb{E}(X^2) = \int_0^{10} x^2\frac{(10-x)^3}{2500}\,dx = \int_0^{10}(10-u)^2\frac{u^3}{2500}\,du = 20/3$, so $\operatorname{Var}(X) = 20/3 - 2^2 = 8/3$.

b. Let $X_1, \ldots, X_{200}$ be the delays of the 200 people. So $\mathbb{E}(X_1 + \cdots + X_{200}) = (200)(2) = 400$ and $\operatorname{Var}(X_1 + \cdots + X_{200}) = (200)(8/3) = 1600/3$. So $P(X_1 + \cdots + X_{200} > 420) = P(\frac{X_1+\cdots+X_{200}-400}{\sqrt{1600/3}} > \frac{420-400}{\sqrt{1600/3}}) \approx P(Z > .87) = 1 - P(Z \leq .87) = 1 - .8078 = .1922$.

4. Let $X_1, \ldots, X_{100}$ be the completion times of the 100 people. So $\mathbb{E}(X_1 + \cdots + X_{100}) = (100)(3.5) = 350$ and $\operatorname{Var}(X_1 + \cdots + X_{100}) = (100)(1/4) = 25$. So $P(348 < X_1 + \cdots + X_{100} < 352) = P(\frac{348-350}{\sqrt{25}} < \frac{X_1+\cdots+X_{100}-350}{\sqrt{25}} > \frac{352-350}{\sqrt{25}}) \approx P(-.4 < Z < .4)$. We break this up as $P(Z < .4) - P(Z \leq -.4) = P(Z < .4) - P(Z \geq .4) = P(Z < .4) - (1 - P(Z < .4)) = 2P(Z < .4) - 1 = (2)(.6554) - 1 = .3108$.

5. We have $\mathbb{E}(X_1 + \cdots + X_{12}) = (12)(0.99) = 11.88$ and $\operatorname{Var}(X_1 + \cdots + X_{12}) = (12)(.03^2) = .0108$. So $P(Y > 1) = P(\frac{X_1+\cdots+X_{12}}{12} > 1) = P(X_1 + \cdots + X_{12} > 12) = P(\frac{X_1+\cdots+X_{12}-11.88}{\sqrt{.0108}} > \frac{12-11.88}{\sqrt{.0108}}) \approx P(Z > 1.15) = 1 - P(Z \leq 1.15) = 1 - .8749 = .1251$.

STAT/MA 41600
Practice Problems #2: November 14, 2014
Solutions by Mark Daniel Ward

1. Let X be the number of flights that are on time. Then X is Binomial with $n = 2000$ and $p = 0.70$, so $P(X > 1420) = P(X > 1420.5) = P\left(\frac{X-(2000)(0.70)}{\sqrt{(2000)(0.70)(0.30)}} > \frac{1420.5-(2000)(0.70)}{\sqrt{(2000)(0.70)(0.30)}}\right) \approx P(Z > 1.00) = 1 - P(Z \leq 1.00) = 1 - .8413 = 0.1587$.

2. Let X be the number of students who attend.

Then X is a Binomial random variable with $n = 400$, $p = 0.60$, so $P(230 \leq X \leq 250) = P(229.5 \leq X \leq 250.5) = P\left(\frac{229.5-(400)(0.60)}{\sqrt{(400)(0.60)(0.40)}} \leq \frac{X-(400)(0.60)}{\sqrt{(400)(0.60)(0.40)}} \leq \frac{250.5-(400)(0.60)}{\sqrt{(400)(0.60)(0.40)}}\right)$. This is approximately $P(-1.07 \leq Z \leq 1.07) = P(Z \leq 1.07) - P(Z < -1.07) = P(Z \leq 1.07) - P(Z > 1.07) = P(Z \leq 1.07) - (1 - P(Z \leq 1.07)) = 2P(Z \leq 1.07) - 1 = 2(.8577) - 1 = .7154$.

3. Let X be the number of broken crayons.

Then X is a Binomial random variable, $n = 12{,}000$, $p = 0.05$, so $P(580 \leq X \leq 620) = P(579.5 \leq X \leq 620.5) = P\left(\frac{579.5-(12{,}000)(0.05)}{\sqrt{(12{,}000)(0.05)(0.95)}} \leq \frac{X-(12{,}000)(0.05)}{\sqrt{(12{,}000)(0.05)(0.95)}} \leq \frac{620.5-(12{,}000)(0.05)}{\sqrt{(12{,}000)(0.05)(0.95)}}\right)$. This is roughly $P(-0.86 \leq Z \leq 0.86) = P(Z \leq 0.86) - P(Z < -0.86) = P(Z \leq 0.86) - P(Z > 0.86) = P(Z \leq 0.86) - (1 - P(Z \leq 0.86)) = 2P(Z \leq 0.86) - 1 = 2(.8051) - 1 = .6102$.

4. Let X be the number of passengers with the extra screening.

Then X is a Binomial random variable with $n = (8)(180) = 1440$ and $p = 0.06$, so $P(X \geq 80) = P(X \geq 79.5) = P\left(\frac{X-(1440)(0.06)}{\sqrt{(1440)(0.06)(0.94)}} \geq \frac{79.5-(1440)(0.06)}{\sqrt{(1440)(0.06)(0.94)}}\right) \approx P(Z \geq -0.77) = P(Z \leq 0.77) = .7794$.

5. Let X be the number of field goals Jeff makes successfully. Let Y be the number of field goals Steve makes successfully. So we want $P(X > Y)$, i.e., $P(X - Y > 0)$. We see that

$$X - Y = X_1 + X_2 + \cdots + X_{120} - Y_1 - Y_2 - \cdots - Y_{164},$$

where X_j indicates whether Jeff's jth attempt was a success, and Y_j indicates whether Steve's jth attempt was a success. So $X - Y$ is the sum of a large number of independent random variables, and thus $X - Y$ is approximately normal.

We have $\mathbb{E}(X - Y) = (120)(.80) - (164)(.60) = -2.40$, and $\operatorname{Var} X - Y = \operatorname{Var} X + \operatorname{Var} Y = (120)(.80)(.20) + (164)(.60)(.40) = 58.56$. Thus $P(X > Y) = P(X - Y > 0) = P(X - Y > 0.5) = P\left(\frac{X-Y-(-2.40)}{\sqrt{58.56}} > \frac{0.5-(-2.40)}{\sqrt{58.56}}\right) \approx P(Z > 0.38) = 1 - P(Z \leq 0.38) = 1 - .6480 = .3520$.

STAT/MA 41600
Practice Problems #3: November 14, 2014
Solutions by Mark Daniel Ward

1. Let X denote the number of Roseate Spoonbills in the 40 hours. Then $P(X \geq 75) = P(X \geq 74.5) = P\left(\frac{X-80}{\sqrt{80}} \geq \frac{74.5-80}{\sqrt{80}}\right) \approx P(Z \geq -0.61) = P(Z \leq 0.61) = 0.7291$.

2. Let X denote the number of errors, so $\mathbb{E}(X) = (6000)(0.04) = 240$ and $\mathrm{Var}(X) = 240$. So $P(X < 230) = P(X < 229.5) = P\left(\frac{X-240}{\sqrt{240}} < \frac{229.5-240}{\sqrt{240}}\right) \approx P(Z < -0.68) = P(Z > 0.68) = 1 - P(Z \leq 0.68) = 1 - 0.7517 = 0.2483$.

3. Let X denote the number of crayons he checks in 40 hours, so $\mathbb{E}(X) = (295)(40) = 11{,}800$ and $\mathrm{Var}(X) = 11{,}800$. So $P(X \geq 12{,}000) = P(X \geq 11{,}999.5) = P\left(\frac{X-11{,}800}{\sqrt{11{,}800}} \geq \frac{11{,}999.5-11{,}800}{\sqrt{11{,}800}}\right) \approx P(Z \geq 1.84) = 1 - P(Z \leq 1.84) = 1 - .9671 = 0.0329$.

4. Let X denote the number of customers, so $\mathbb{E}(X) = (8)(168) = 1344$ and $\mathrm{Var}(X) = 1344$. So $P(1300 \leq X \leq 1400) = P(1299.5 \leq X \leq 1400.5) = P\left(\frac{1299.5-1344}{\sqrt{1344}} \leq \frac{X-1344}{\sqrt{1344}} \leq \frac{1400.5-1344}{\sqrt{1344}}\right) \approx P(-1.21 \leq Z \geq 1.54) = P(Z \leq 1.54) - P(Z < -1.21) = P(Z \leq 1.54) - P(Z > 1.21) = P(Z \leq 1.54) - (1 - P(Z \leq 1.21)) = .9382 - (1 - .8869) = .8251$.

5. Let X denote the number of Dr. Ward's errors, and let Y denote the number of his wife's errors. As in question #2, we have $\mathbb{E}(X) = (6000)(0.04) = 240$ and $\mathrm{Var}(X) = 240$. Also $\mathbb{E}(Y) = (10{,}000)(0.025) = 250$ and $\mathrm{Var}(Y) = 250$. So $P(X > Y) = P(X - Y > 0) = P(X - Y > 0.5) = P\left(\frac{X-Y-(240-250)}{\sqrt{240+250}} > \frac{0.5-(240-250)}{\sqrt{240+250}}\right) \approx P(Z > 0.47) = 1 - P(Z \leq 0.47) = 1 - 0.6808 = 0.3192$.

PurdueX: 416.2x
Probability: Distribution Models & Continuous Random Variables

Solution to the problem sets

Unit 11: Covariance, Conditional Expectation, Markov and Chebychev Inequalities

STAT/MA 41600
Practice Problems: November 24, 2014
Solutions by Mark Daniel Ward

1. *Method #1:* Since X is hypergeometric with $M = 8$, $N = 11$, and $n = 2$, then $\mathrm{Var}(X) = n\frac{M}{N}\left(1 - \frac{M}{N}\right)\frac{N-n}{N-1} = 2\frac{8}{11}\left(1 - \frac{8}{11}\right)\frac{11-2}{11-1} = 216/605$.

Method #2: The mass of X is $p_X(0) = \binom{3}{2}/\binom{11}{2} = 3/55$; $p_X(1) = \binom{3}{1}\binom{8}{1}/\binom{11}{2} = 24/55$; $p_X(2) = \binom{8}{2}/\binom{11}{2} = 28/55$. Thus $\mathbb{E}(X) = (0)(3/55) + (1)(24/55) + (2)(28/55) = 16/11$, and $\mathbb{E}(X^2) = (0^2)(3/55) + (1^2)(24/55) + (2^2)(28/55) = 136/55$, so $\mathrm{Var}(X) = 136/55 - (16/11)^2 = 216/605$.

Method #3: Using the methods of Chapter 42, we write X_1 to indicate if Alice gets lemonade, and X_2 to indicate if Bob gets lemonade. So X_1 and X_2 are dependent Bernoulli's. Thus $\mathbb{E}(X_1) = \mathbb{E}(X_2) = 8/11$, and $\mathrm{Var}(X_1) = \mathrm{Var}(X_2) = (8/11)(3/11) = 24/121$. Also $\mathrm{Var}(X) = \mathrm{Var}(X_1 + X_2) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + 2\,\mathrm{Cov}(X_1, X_2)$. We know $\mathrm{Cov}(X_1, X_2) = \mathbb{E}(X_1X_2) - \mathbb{E}(X_1)\mathbb{E}(X_2) = \mathbb{E}(X_1X_2) - (8/11)(8/11) = \mathbb{E}(X_1X_2) - 64/121$. Also, X_1X_2 is 0 or 1, so X_1X_2 is Bernoulli, so $\mathbb{E}(X_1X_2) = P(X_1X_2 = 1) = P(X_1 = 1 \text{ and } X_2 = 1) = P(X_1 = 1)P(X_2 = 1 \mid X_1 = 1) = (8/11)(7/10) = 28/55$. So, altogether, we have $\mathrm{Var}(X) = 24/121 + 24/121 + 2(28/55 - 64/121) = 216/605$.

2. As in the "Method #3" solution from question 1, we have $\mathrm{Cov}(X_1, X_2) = \mathbb{E}(X_1X_2) - \mathbb{E}(X_1)\mathbb{E}(X_2) = 28/55 - (8/11)(8/11) = -12/605$. Also $\mathrm{Var}(X_1) = \mathrm{Var}(X_2) = 24/121$. Thus $\rho(X_1, X_2) = \frac{\mathrm{Cov}(X_1,X_2)}{\sqrt{\mathrm{Var}(X_1)\,\mathrm{Var}(X_2)}} = \frac{-12/605}{\sqrt{(24/121)(24/121)}} = -1/10$.

3a. We have $\mathrm{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$. Since X is uniform on $[10, 14]$, then $\mathbb{E}(X) = 12$. Since Y is uniform on $[22, 30]$, then $\mathbb{E}(Y) = 26$. Also $\mathbb{E}(XY) = \int_{10}^{14} x(2x+2)\frac{1}{4}\,dx = 944/3$. Thus $\mathrm{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 944/3 - (12)(26) = 8/3$.

b. Since X is uniform on $[10, 14]$, then $\mathrm{Var}(X) = (14 - 10)^2/12 = 4/3$. Since Y is uniform on $[22, 30]$, then $\mathrm{Var}(Y) = (30 - 22)^2/12 = 16/3$. So $\rho(X, Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} = \frac{8/3}{\sqrt{(4/3)(16/3)}} = 1$.

4a. We have $\mathrm{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$. Since X is uniform on $[3, 6]$, then $\mathbb{E}(X) = 4.5$. We already calculated $\mathbb{E}(Y) = 20$ in part c and part d of question 3 on problem set 35. Finally, we need $\mathbb{E}(XY) = \int_3^6 x(x^2 - 1)\frac{1}{3}\,dx = \int_3^6 x(x^2 - 1)\frac{1}{3}\,dx = 387/4$. Thus $\mathrm{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 387/4 - (4.5)(20) = 27/4$.

b. Since X is uniform on $[3, 6]$, then $\mathrm{Var}(X) = (6 - 3)^2/12 = 3/4$.
Also $\mathbb{E}(Y^2) = \int_3^6 (x^2 - 1)^2(1/3)\,dx = 2306/5$. So $\mathrm{Var}(Y) = 2306/5 - 20^2 = 306/5$.
Thus $\rho(X, Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} = \frac{27/4}{\sqrt{(3/4)(306/5)}} = 0.996$.

5. The mass of X is $p_X(1) = 7/16$; $p_X(2) = 5/16$; $p_X(3) = 3/16$; $p_X(4) = 1/16$; so $\mathbb{E}(X) = (1)(7/16) + (2)(5/16) + (3)(3/16) + (4)(4/16) = 15/8$.

The mass of Y is $p_Y(1) = 1/16$; $p_Y(2) = 3/16$; $p_Y(3) = 5/16$; $p_Y(4) = 7/16$; so $\mathbb{E}(Y) = (1)(1/16) + (2)(3/16) + (3)(5/16) + (4)(7/16) = 25/8$.

The expected value of XY is

$$\begin{aligned}\mathbb{E}(XY) &= \frac{1}{16}((1)(1) + (1)(2) + (1)(3) + (1)(4)\\ &\quad + (1)(2) + (2)(2) + (2)(3) + (2)(4)\\ &\quad + (1)(3) + (2)(3) + (3)(3) + (3)(4)\\ &\quad + (1)(4) + (2)(4) + (3)(4) + (4)(4))\\ &= (1/16)(1 + 2 + 3 + 4 + 2 + 4 + 6 + 8 + 3 + 6 + 9 + 12 + 4 + 8 + 12 + 16)\\ &= 25/4\end{aligned}$$

Thus $\mathrm{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 25/4 - (15/8)(25/8) = 25/64$.

STAT/MA 41600
Practice Problems: December 1, 2014
Solutions by Mark Daniel Ward

1. *Method #1:* Since X, Y have a joint uniform distribution on the triangle, then given $X = 1/2$, we know that Y is uniformly distributed on $[0, 3/2]$. Thus the conditional expectation of Y given $X = 1/2$ is exactly $\mathbb{E}(Y \mid X = 1/2) = \frac{3/2+0}{2} = 3/4$.

Method #2: The area of the triangle is 2, so $f_{X,Y}(x,y) = 1/2$ on the triangle. Also $f_X(1/2) = \int_0^{3/2} f_{X,Y}(x,y)\,dy = \int_0^{3/2} 1/2\,dy = 3/4$. So $f_{Y|X}(y \mid 2) = \frac{f_{X,Y}(2,y)}{f_X(2)} = \frac{1/2}{3/4} = 2/3$. Thus $\mathbb{E}(Y \mid X = 2) = \int_0^{3/2}(y)(2/3)\,dy = 3/4$.

2. a. *Method #1:* Given $X = 3$, there are 7 equally-likely outcomes: $(3,3),(3,4),(3,5),(3,6),(4,3),(5,3),(6,3)$. So $\mathbb{E}(Y \mid X = 3) = \frac{1}{7}(3+4+5+6+4+5+6) = \frac{33}{7}$.

Method #2: We have $p_X(3) = 7/36$. Also $p_{X,Y}(3,3) = 1/36$, and $p_{X,Y}(3,y) = 2/36$ for $y = 4,5,6$. Also $p_{Y|X}(y \mid 3) = \frac{p_{X,Y}(3,y)}{p_X(3)}$. So $p_{Y|X}(y \mid 3) = \frac{1/36}{7/36} = 1/7$, and $p_{Y|X}(y \mid 3) = \frac{2/36}{7/36} = 2/7$, for $y = 4,5,6$. So $\mathbb{E}(Y \mid X = 3) = (3)(1/7) + (4)(2/7) + (5)(2/7) + (6)(2/7) = 33/7$.

b. We have $\mathbb{E}(X + Y \mid X = 3) = \mathbb{E}(X \mid X = 3) + \mathbb{E}(Y \mid X = 3) = 3 + 33/7 = 54/7$.

3. We know that Y is a Gamma random variable with $\lambda = 1$ and $r = 2$. Thus $f_Y(3) = e^{-3}3/(2!) = 3e^{-3}$. Also $f_{X_1,Y}(x,3) = f_{Y|X_1}(3 \mid x)f_{X_1}(x)$. Of course $f_{X_1}(x) = e^{-x}$. For any $y > X_1$, we have $F_{Y|X_1}(y \mid x) = P(Y < y \mid X_1 = x) = P(Y - x < y - x \mid X_1 = x) = P(X_2 < y - x) = F_{X_2}(y-x)$. Differentiating with respect to y gives $f_{Y|X_1}(y \mid x) = f_{X_2}(y-x) = e^{-(y-x)}$. So $f_{Y|X_1}(3 \mid x) = e^{-(3-x)}$ for $3 > x$. So $f_{X_1,Y}(x,3) = f_{Y|X_1}(3 \mid x)f_{X_1}(x) = e^{-(3-x)}e^{-x} = e^{-3}$. So $f_{X_1 \mid Y}(x_1 \mid 3) = \frac{f_{X_1,Y}(x,3)}{f_Y(3)} = \frac{e^{-3}}{3e^{-3}} = 1/3$. Thus the conditional density of X_1, given $Y = 3$, is uniform on $[0,3]$. So $\mathbb{E}(X_1 \mid Y = 3) = 3/2$. I.e., $\mathbb{E}(X_1 \mid Y = 3) = \int_0^3 (x)(1/3)\,dx = 3/2$.

4. a. If Bob gets lemonade, then Alice has 10 remaining drinks, of which 7 are lemonade, so $\mathbb{E}(X_1 \mid X_2 = 1) = P(X_1 = 1 \mid X_2 = 1) = 7/10$.

b. If Bob does not get lemonade, then Alice has 10 remaining drinks, of which 8 are lemonade, so $\mathbb{E}(X_1 \mid X_2 = 0) = P(X_1 = 1 \mid X_2 = 0) = 8/10$.

5. *Method #1:* Given that there are 12 roses, then each is equally likely to have been picked by Sally or David, so for each flower, we expect it was picked by Sally half the time or by David half the time. So the expected number of roses picked by Sally is 6.

More formally, to see the argument in Method #1, let $X_1, \ldots, X_{10}$ be indicators for whether the 1st, 2nd, ..., 10th flower of Sally is a rose. Then $X = X_1 + \cdots + X_{10}$, so

$$\begin{aligned}\mathbb{E}(X \mid Y = 12) &= \mathbb{E}(X_1 + \cdots + X_{10} \mid Y = 12) \\ &= \mathbb{E}(X_1 \mid Y = 12) + \cdots + \mathbb{E}(X_{10} \mid Y = 12) \\ &= 12/20 + \cdots + 12/20 \\ &= (10)(12/20) \\ &= 6.\end{aligned}$$

Method #3: We can go through the same kind of argument as in the cookie example in the Conditional Expectation chapter of the book, using 10 flowers per person instead of 5 cookies per person, and using $Y = 12$ instead of $Y = 7$. We will get $p_{X|Y}(x \mid 12) = \frac{\binom{10}{x}\binom{10}{12-x}}{\binom{20}{12}}$. So, conditioned on $Y = 12$, we see that X is hypergeometric with $M = 10$ flowers for Sally, and $N = 20$ flowers altogether, and $n = 12$ of the flowers are selected to be designated as roses. Thus $\mathbb{E}(X \mid Y = 12) = nM/N = (12)(10)/20 = 6$.

STAT/MA 41600
Practice Problems: December 3, 2014
Solutions by Mark Daniel Ward

1a. Let X be the studying time. Then $P(X \geq 7) \leq \mathbb{E}(X)/7 = 5/7$.

1b. We have $P(3 \leq X \leq 7) = P(|X-5| \leq 2)$, but $2 = (8/5)(5/4)$, so $P(3 \leq X \leq 7) = P(|X-5| \leq (8/5)(5/4)) \geq \frac{(8/5)^2-1}{(8/5)^2} = 39/64$.

2. Let X the time between two consecutive sneezes. Then $\mathbb{E}(X) = 35$ and $\sigma_X = 1.5$. So $P(30 \leq X \leq 40) = P(|X-35| \leq 5)$, but $5 = (10/3)(3/2)$, so $P(30 \leq X \leq 40) = P(|X-35| \leq (10/3)(3/2)) \geq \frac{(10/3)^2-1}{(10/3)^2} = 91/100$.

3. a. Let X be the amount of food eaten. Then $P(X \geq 1000) \leq \mathbb{E}(X)/1000 = 750/1000 = 3/4$.

b. We have $P(X > 1000 \text{ or } X < 500) = P(|X-750| \geq 250)$, but $250 = (250/100)(100)$, so $P(X > 1000 \text{ or } X < 500) = P(|X-750| \geq (250/100)(100)) \leq \frac{1}{(250/100)^2} = 4/25$.

4. a. Let X be the number of people needed to find the 25th person who likes artichokes. Then X is Negative Binomial with $r = 25$ and $p = .11$. So $\mathbb{E}(X) = 25/(.11) = 2500/11 = 227.27$.

b. Since X is Negative Binomial with $r = 25$ and $p = .11$ and $q = 1 - p = .89$, then $\text{Var}(X) = qr/p^2 = 222500/121 = 1838.84$.

5. a. The random variable Y is a Gamma random variable with $r = 2$ and $\lambda = 1/10$.

b. We have $\mathbb{E}(Y) = r/\lambda = (2)(10) = 20$.

c. We have $\text{Var}\, Y = r/\lambda^2 = (2)(10^2) = 200$.

d. The density of Y is $f_Y(y) = \frac{(1/10)^2}{\Gamma(2)} y^{2-1} e^{-y/10} = \frac{ye^{-y/10}}{100}$ for $y > 10$, and $f_Y(y) = 0$ otherwise. So $P(Y > 12) = \int_{12}^{\infty} f_Y(y)\,dy = \frac{11}{5}e^{-6/5} = .6626$.

PurdueX: 416.2x
Probability: Distribution Models & Continuous Random Variables

Solution to the problem sets

Unit 12: Order Statistics, Moment Generating Functions, Transformation of RVs

STAT/MA 41600
Practice Problems: December 5, 2014
Solutions by Mark Daniel Ward

1. a. For $0 \le x_1 \le 20$, we have $f_{X_{(1)}}(x_1) = \binom{4}{0,1,3}\left(\frac{1}{20}\right)\left(\frac{x_1}{20}\right)^0\left(1-\frac{x_1}{20}\right)^3 = (4)\left(\frac{1}{20}\right)^4(20-x_1)^3$. Otherwise, $f_{X_{(1)}}(x_1) = 0$.

b. For $0 \le x_2 \le 20$, we have $f_{X_{(2)}}(x_2) = \binom{4}{1,1,2}\left(\frac{1}{20}\right)\left(\frac{x_2}{20}\right)^1\left(1-\frac{x_2}{20}\right)^2 = (12)\left(\frac{1}{20}\right)^4(x_2)(20-x_2)^2$. Otherwise, $f_{X_{(2)}}(x_2) = 0$.

2. a. We integrate using **1a** and get $\mathbb{E}(X_{(1)}) = \int_0^{20}(x_1)(4)\left(\frac{1}{20}\right)^4(20-x_1)^3\,dx_1 = 4$.

b. We integrate using **1b** and get $\mathbb{E}(X_{(2)}) = \int_0^{20}(x_2)(12)\left(\frac{1}{20}\right)^4(x_2)(20-x_2)^2\,dx_2 = 8$.

3. a. For $0 < x_1$, we have $f_{X_{(1)}}(x_1) = \binom{2}{0,1,1}\frac{1}{10}e^{-x_1/10}\left(1-e^{-x_1/10}\right)^0\left(e^{-x_1/10}\right)^1 = \frac{2}{10}(e^{-x_1/10})^2$. Otherwise, $f_{X_{(1)}}(x_1) = 0$.

b. For $0 < x_2$, we have $f_{X_{(2)}}(x_2) = \binom{2}{1,1,0}\frac{1}{10}e^{-x_2/10}\left(1-e^{-x_2/10}\right)^1\left(e^{-x_2/10}\right)^0 = \frac{2}{10}(e^{-x_2/10})(1-e^{-x_2/10})$. Otherwise, $f_{X_{(2)}}(x_2) = 0$.

4. a. We integrate using **3a** and get $\mathbb{E}(X_{(1)}) = \int_0^{\infty}(x_1)(\frac{2}{10})(e^{-x_1/10})^2\,dx_1 = 5$.

b. We integrate using **3b** and get $\mathbb{E}(X_{(2)}) = \int_0^{\infty}(x_2)(\frac{2}{10})(e^{-x_2/10})(1-e^{-x_2/10})\,dx_2 = 15$.

5. a. The CDF of each of the random variables is, for $0 < a < 1$,

$$F_X(a) = P(X \le a) = \int_0^a 6(x-x^2)\,dx = -2a^3 + 3a^2.$$

Thus for $0 < x_1 < 1$,

$$\begin{aligned} f_{X_{(1)}}(x_1) &= \binom{2}{0,1,1} f_X(x_1)(1-F_X(x_1)) \\ &= \binom{2}{0,1,1}\left(6(x_1-x_1^2)\right)\left(1+2x_1^3-3x_1^2\right) \\ &= 12x_1 - 12x_1^2 - 36x_1^3 + 60x_1^4 - 24x_1^5 \end{aligned}$$

Otherwise, $f_{X_{(1)}}(x_1) = 0$.

b. We integrate using **5a** and get $\mathbb{E}(X_{(1)}) = \int_0^1(x_1)(12x_1 - 12x_1^2 - 36x_1^3 + 60x_1^4 - 24x_1^5)\,dx_1 = 13/35$.

STAT/MA 41600
Practice Problems: December 10, 2014
Solutions by Mark Daniel Ward

1. a. Since X is uniform and Y has the form $Y = aX + b$ for constants a, b, then Y is uniform too. Notice $2(10) + 2 \leq Y \leq 2(14) + 2$, i.e., $22 \leq Y \leq 30$. Also $\frac{1}{30-22} = \frac{1}{8}$. So $f_Y(y) = \frac{1}{8}$ for $22 \leq Y \leq 30$, and $f_Y(y) = 0$ otherwise.

b. Since the density of Y is constant on $[22, 30]$, then $P(Y > 28) = \frac{\text{length of } [28,30]}{\text{length of } [22,30]} = 2/8 = 1/4$.

c. We have $P(Y > 28) = P(2X + 2 > 28) = P(2X > 26) = P(X > 13)$. Since the density of X is constant on $[10, 14]$, then $P(X > 13) = \frac{\text{length of } [13,14]}{\text{length of } [10,14]} = 1/4$.

2. a. Since X is uniform and Y has the form $Y = aX + b$ for constants a, b, then Y is uniform too. Notice $(1.07)(4) + 3.99 \leq Y \leq (1.07)(9) + 3.99$, i.e., $8.27 \leq Y \leq 13.62$. Also $\frac{1}{13.62-8.27} = \frac{1}{5.35}$. So $f_Y(y) = \frac{1}{5.35}$ for $8.27 \leq Y \leq 13.62$, and $f_Y(y) = 0$ otherwise.

b. *Method #1:* Since Y is uniform on $[8.27, 13.62]$, the expected value of Y is the midpoint of the interval, i.e., $\mathbb{E}(Y) = \frac{8.27+13.62}{2} = 10.945$.

Method #2: We calculate $\mathbb{E}(Y) = \int_{8.27}^{13.62} y \frac{1}{5.35}\, dy = \frac{13.62^2 - 8.27^2}{2} \frac{1}{5.35} = 10.945$.

c. We calculate $\mathbb{E}(X) = \int_4^9 (1.07x + 3.99)\frac{1}{5}\, dx = \frac{1}{5}(1.07x^2/2 + 3.99x)|_{x=4}^{9} = 10.945$.

3. a. For $8 \leq a \leq 35$, we have $P(Y \leq a) = P((X - 1)(X + 1) \leq a) = P(X^2 - 1 \leq a) = P(X^2 \leq a + 1) = P(X \leq \sqrt{a + 1}) = \frac{\sqrt{a+1}-3}{6-3}$. Thus, the CDF of Y is

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 8, \\ \frac{\sqrt{y+1}-3}{3} & \text{if } 8 \leq y \leq 35, \\ 1 & \text{if } 35 < y. \end{cases}$$

b. For $8 \leq y \leq 35$, we differentiate $F_Y(y)$ with respect to y, and we get $f_Y(y) = \frac{1}{6}(y+1)^{-1/2}$; otherwise, $f_Y(y) = 0$.

c. We use $u = y + 1$ and $du = dy$ to compute $\mathbb{E}(Y) = \int_8^{35} y\, \frac{1}{6}(y + 1)^{-1/2}\, dy = \int_9^{36} \frac{1}{6}(u - 1)u^{-1/2}\, du = \int_9^{36} \frac{1}{6}(u^{1/2} - u^{-1/2})\, du = \frac{1}{6}(\frac{2}{3}u^{3/2} - 2u^{1/2})|_{u=9}^{36} = \frac{1}{6}(((2/3)(216) - (2)(6)) - ((2/3)(27) - (2)(3))) = 20$.

d. We compute $\mathbb{E}((X - 1)(X + 1)) = \int_3^6 (x - 1)(x + 1)(1/3)\, dx = \int_3^6 (1/3)(x^2 - 1)\, dx = (1/3)(\frac{1}{3}x^3 - x)|_{x=3}^{6} = (1/3)((72 - 6) - (9 - 3)) = 20$.

4. We have $\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$. Since X, Y have a joint uniform distribution on a triangle with area $(2)(2)/2 = 2$, then $f_{X,Y}(x, y) = 1/2$ on the triangle, and $f_{X,Y}(x, y) = 0$

otherwise. So:

$$\mathbb{E}(XY) = \int_0^2 \int_0^{2-x} xy\frac{1}{2}\,dy\,dx = 1/3,$$

and

$$\mathbb{E}(X) = \int_0^2 \int_0^{2-x} x\frac{1}{2}\,dy\,dx = 2/3,$$

and (since everything is symmetric, we don't even need to calculate):

$$\mathbb{E}(Y) = \int_0^2 \int_0^{2-x} y\frac{1}{2}\,dy\,dx = 2/3.$$

So $\mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 1/3 - (2/3)(2/3) = -1/9$.

5. Since X_j is Bernoulli with $p = 2/19$, then $\mathrm{Var}(X_j) = (2/19)(17/19) = 34/361$.

Also $\mathrm{Cov}(X_i, X_j) = \mathbb{E}(X_iX_j) - \mathbb{E}(X_i)\mathbb{E}(X_j)$. Also $\mathbb{E}(X_i) = 2/19$ and $\mathbb{E}(X_j) = 2/19$, so we only need $\mathbb{E}(X_iX_j)$. Notice X_iX_j is 0 or 1, i.e., the product X_iX_j is Bernoulli, so $\mathbb{E}(X_iX_j) = P(X_iX_j = 1)$. [We can also see this by $\mathbb{E}(X_iX_j) = 1P(X_iX_j = 1) + 0P(X_iX_j = 0) = P(X_iX_j = 1)$.]

Now we use $P(X_iX_j = 1) = P(X_i = 1 \text{ and } X_j = 1) = P(X_i = 1)P(X_j = 1 \mid X_i = 1)$. We know $P(X_i = 1) = 2/19$. Once $X_i = 1$ is given, there is a row of 18 open seats where the jth couple might sit. The man sits on the end with probability 2/18 and his wife beside him with probability 1/17, or the man does not sit on the end, with probability 16/18 and his wife beside him with probability 2/17, so $P(X_j = 1 \mid X_i = 1) = (2/18)(1/17) + (16/18)(2/17) = 1/9$. [Alternatively, this can be calculated by observing that there are (18)(17) places that they can sit, but there are 17 adjacent pairs of seats, and they can sit in them 2 ways, so $P(X_j = 1 \mid X_i = 1) = \frac{(17)(2)}{(18)(17)} = 2/18 = 1/9$.]

So $\mathbb{E}(X_iX_j) = P(X_iX_j = 1) = (2/19)(1/9)$.

So $\mathrm{Cov}(X_i, X_j) = \mathbb{E}(X_iX_j) - \mathbb{E}(X_i)\mathbb{E}(X_j) = (2/19)(1/9) - (2/19)(2/19) = 2/3249$.

Finally $\mathrm{Var}(X) = \sum_{j=1}^{10} \mathrm{Var}(X_j) + 2\sum\sum_{1\le i<j\le 10} \mathrm{Cov}(X_i, X_j) = (10)(34/361) + (90)(2/3249) = 360/361$.