PurdueX: 416.1x
Probability: Basic Concepts & Discrete Random Variables

Problem sets

Unit 1: Sample Space and Probability

STAT/MA 41600
Practice Problems: August 27, 2014

1. Choosing points at random.
(a.) A point is chosen at random inside in the triangle in Figure 1. What is the sample space? [Please give mathematical expression(s), rather than just (x, y) is "in the triangle".]

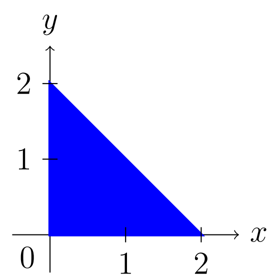

Figure 1: A triangle.

(b.) A point is chosen at random in the quadrilateral in Figure 2. What is the sample space? [Hint: It might be helpful to give bounds on the x coordinate and then give bounds on the y coordinate.]

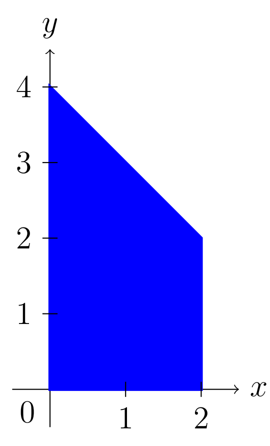

Figure 2: A quadrilateral.

2. Gloves. A matching pair of blue gloves, a matching pair of red gloves, and one lone white right-handed glove are in a drawer. The gloves are pulled out of the drawer, one at a time.

Suppose the color and the "hand" of the gloves are both noted as all five gloves are consecutively pulled out of the drawer. Then, for instance, a possible outcome is:

("blue right", "white right", "red left", "red right", "blue left").

(a.) How many outcomes are in the sample space?

(b.) Now suppose that only the color (not the hand) of the glove is noted as the gloves are removed. This drastically shrinks the size of the sample space. For instance, one possible outcome would be:

("blue", "white", "red", "red", "blue").

Now how many outcomes are in the sample space?

3. Seating arrangements. Alice, Bob, Catherine, Doug, and Edna are randomly assigned seats at a circular table in a perfectly circular room. Assume that rotations of the table do not matter, so there are exactly 24 possible outcomes in the sample space.

Bob and Catherine are married. Doug and Edna are married. When people are married they love to sit beside each other. In how many of these 24 outcomes are both married couples sitting together and therefore happy?

4. Abstract art. A painter has three different jars of paint colors available, namely, green, yellow, and purple. She wants to paint something abstract, so she blindfolds herself, randomly dips her brush, and paints on the canvas. She continues trying paint jars until she finally gets some purple onto the canvas (her assistant will tell her when this happens) and then she stops. Assume that she does not repeat any of the jars because her assistant removes a jar once it has been used. So the sample space is

$$S = \{(P), (G, P), (Y, P), (Y, G, P), (G, Y, P)\}.$$

How many events can be made using this sample space?

5. Sum of three dice. Roll three distinguishable dice (e.g., assume that there is a way to tell them apart, for instance, that the dice are three different colors). There are $6 \times 6 \times 6 = 216$ possible outcomes.

For $3 \leq j \leq 18$, define A_j as the event that the sum of the dice equals j. How many outcomes are in each A_j? (For instance, A_3 contains only one outcome, namely, $(1, 1, 1)$. Similarly, A_{18} contains only one outcome, namely, $(6, 6, 6)$. These are filled in already, in the table below. Your table entries should sum to 216 altogether.)

event	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
number of outcomes	1							

event	A_{11}	A_{12}	A_{13}	A_{14}	A_{15}	A_{16}	A_{17}	A_{18}
number of outcomes								1

Hint: It might be helpful to think about the 11 possible events that are possible with just 2 dice, described here:

x	2	3	4	5	6	7	8	9	10	11	12
$P(X = x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

STAT/MA 41600

Practice Problems: August 29, 2014

1. Choosing a page at random.

A student buys a brand new calculus textbook that has 1000 pages, each numbered with 3 digits, from 000 to 999. She randomly opens the book to a page and starts to read! Assume that any of the 1000 pages are equally likely to be chosen.

Find the probability that the page number she chooses contains at least one "5" as a digit. (For instance, each page in the range 500 to 599 contains a "5", of course, at the start. Also, page 605 has a "5", and page 257 contains a "5", and page 055 contains a "5", etc.) Hint: Use inclusion-exclusion.

2. Gloves. A matching pair of blue gloves, a matching pair of red gloves, and one lone white right-handed glove are in a drawer. The gloves are pulled out of the drawer, one at a time.

(a.) Suppose that a person is looking for the white glove. He repeatedly does the following: He pulls out a glove, checks the color, and if it is white, he stops. If it is not white, then he replaces the glove in the drawer and starts to check again, i.e., he reaches blindly into the drawer of 5 gloves. He continues to do this over and over until he finds the white glove, and then he stops. Let A_j denote the event that he successfully discovers the white glove (for the first time) on his jth attempt. Find $P(A_j)$.

(b.) Now suppose that he searches for the white glove but, if he pulls a different colored glove from the drawer, he does not replace it. So he pulls out a glove, checks the color, and if it is white, he stops. If it is not white, then he *permanently discards* the glove and starts to check again, i.e., he reaches blindly into the drawer with the gloves that remain. He continues to do this over and over again until he finds the white glove, and then he stops. Let B_j denote the event that he successfully discovers the white glove (for the first time) on his jth attempt. Find $P(B_1), P(B_2), P(B_3), P(B_4)$, and $P(B_5)$.

3. Seating arrangements. Alice, Bob, Catherine, Doug, and Edna are randomly assigned seats at a circular table in a perfectly circular room. Assume that rotations of the table do not matter, so there are exactly 24 possible outcomes in the sample space.

Bob and Catherine are married. Doug and Edna are married. When people are married they love to sit beside each other.

Let A_j denote the event that exactly j of the married couples are happy because they are sitting together. Find $P(A_0)$ and $P(A_1)$ and $P(A_2)$.

4. Abstract art. A painter has three different jars of paint colors available, namely, green, yellow, and purple. She wants to paint something abstract, so she blindfolds herself, randomly dips her brush, and paints on the canvas. She continues trying paint jars until she finally gets some purple onto the canvas (her assistant will tell her when this happens) and then she stops. Assume that she does not repeat any of the jars because her assistant removes a jar once it has been used. So the sample space is

$$S = \{(P), (G, P), (Y, P), (Y, G, P), (G, Y, P)\}.$$

Find the probabilities of each of the following events:

EVENT:	Probability of the event:
$\{(P)\}$	
$\{(G, P), (Y, P)\}$	
$\{(G, P), (G, Y, P)\}$	
$\{(Y, G, P), (G, Y, P)\}$	
$\{(P), (Y, P)\}$	

5. Maximum of three dice. Roll three distinguishable dice (e.g., assume that there is a way to tell them apart, for instance, that the dice are three different colors). There are $6 \times 6 \times 6 = 216$ possible outcomes.

Let B_k be the event that the maximum value that appears on all three dice when they are rolled is less than or equal to k. Find $P(B_1), P(B_2), P(B_3), P(B_4), P(B_5)$, and $P(B_6)$. If you prefer, you are welcome to just give a general formula that covers all six of these cases, i.e., you are welcome to just give a formula for $P(B_k)$ itself.

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Probability: Basic Concepts & Discrete Random Variables

Problem sets

Unit 2: Independent Events, Conditional Probability and Bayes' Theorem

STAT/MA 41600
Practice Problems: September 3, 2014

1. Choosing a page at random.

A student buys a brand new calculus textbook that has 1000 pages, each numbered with 3 digits, from 000 to 999. She randomly opens the book to a page and starts to read! Assume that any of the 1000 pages are equally likely to be chosen.

Let A be the event that the first digit on the chosen page is a “5” (for example, the page could be 572). Let B be the event that the second and third digits on the chosen pages are both “5” (for example, the page could be 455).

Are A and B independent? Why? Justify your answer.

2. Gloves. A matching pair of blue gloves, a matching pair of red gloves, and one lone white right-handed glove are in a drawer. The gloves are pulled out of the drawer, one at a time.

Suppose that a person is looking for the white glove. He repeatedly does the following: He pulls out a glove, checks the color, and if it is white, he stops. If it is not white, then he replaces the glove in the drawer and starts to check again, i.e., he reaches blindly into the drawer of 5 gloves. He continues to do this over and over until he finds the white glove, and then he stops.

Let A denote the event that he pulls out a red glove during this process. In other words, A denotes the event that he finds a red glove before a white glove. Find $P(A)$.

3. Seating arrangements. Alice, Bob, Catherine, Doug, and Edna are randomly assigned seats at a circular table in a perfectly circular room. Assume that rotations of the table do not matter, so there are exactly 24 possible outcomes in the sample space.

Bob and Catherine are married. Doug and Edna are married. When people are married they love to sit beside each other.

Let T denote the event that Bob and Catherine are sitting next to each other. Let U be the event that Alice and Bob are sitting next to each other. Are events T and U independent? Why? Justify your answer.

4. Abstract art. A painter has three different jars of paint colors available, namely, green, yellow, and purple. She wants to paint something abstract, so she blindfolds herself, randomly dips her brush, and paints on the canvas. She continues trying paint jars **until she has used all three of them** (this is different than the story from the last two days), and then she stops. Assume that she does not repeat any of the jars because her assistant removes a jar once it has been used. So the sample space is

$$S = \{(G, P, Y), (G, Y, P), (P, G, Y), (P, Y, G), (Y, G, P), (Y, P, G)\}.$$

Let A be the event that purple is found in the second jar tested by the painter. Let B be the event that green is found before yellow. Are events A and B independent? Why? Justify your answer.

5. Even versus four or less. Roll a die. Let A be the event that the outcome on the die is an even number. Let B be the event that the outcome on the die is 4 or smaller. Let C be the event that the outcome on the die is 3 or larger.

Are A and B independent? Why? Justify your answer.

Are B and C independent? Why? Justify your answer.

STAT/MA 41600
Practice Problems: September 5, 2014

1. Choosing a page at random.

A student buys a brand new calculus textbook that has 1000 pages, each numbered with 3 digits, from 000 to 999. She randomly opens the book to a page and starts to read! Assume that any of the 1000 pages are equally likely to be chosen.

Let B be the event that *at least one* of the digits on the chosen page is a "5" (for example, the page could be 572 or 505 or 355 or 115 etc., etc.). Let C be the event that *at least two* of the digits on the chosen page are "5" (for example, 355 or 575, etc.).

(a.) Given that B occurs, find the conditional probability of A, the event that 555 is the selected page.

(b.) Given that C occurs, find the conditional probability of A, the event that 555 is the selected page.

2. Gloves. A matching pair of blue gloves, a matching pair of red gloves, and one lone white right-handed glove are in a drawer.

Suppose that a person is looking for the two matching blue gloves. He blindly pulls two gloves out of the drawer simultaneously. Let B be the event that *at least one* of the gloves is blue.

(a.) Given that B occurs, find the conditional probability of A, the event that both of the gloves are blue.

(b.) Given that B occurs, find the conditional probability of A^c.

3. Seating arrangements. Alice, Bob, Catherine, Doug, and Edna are randomly assigned seats at a circular table in a perfectly circular room. Assume that rotations of the table do not matter, so there are exactly 24 possible outcomes in the sample space.

Bob and Catherine are married. Doug and Edna are married. When people are married they love to sit beside each other.

Let B denote the event that Bob and Catherine are sitting next to each other. Given that B occurs, find the conditional probability of A, the event that Doug and Edna are sitting next to each other.

4. Pair of dice. Roll a pair of dice. Let B be the event that the two dice have different values. Given that B occurs, find the conditional probability of A, the event that the sum of the dice is an even number.

5. Pair of dice. Roll a pair of dice. Let B be the event that the sum of the pair of dice is 9 or larger. Given that B occurs, find the conditional probability of A, the event that the sum of the pair of dice is exactly 10.

STAT/MA 41600
Practice Problems: September 8, 2014

1. Waking up at random.

On each weekday, a student wakes up before 8 AM with probability .65, or after 8 AM with probability .35.

On each weekend, a student wakes up before 8 AM with probability .22, or after 8 AM with probability .78.

Suppose that a student wakes up and can't remember which day it is!

(a.) If the student sees that it is before 8 AM, what is the probability it is a weekday?

(b.) If the student sees that it is after 8 AM, what is the probability it is a weekday?

2. Gloves. A matching pair of blue gloves, a matching pair of red gloves, and one lone white right-handed glove are in a drawer.

Suppose that a person blindly pulls a glove out of the drawer and it is not red (i.e., it is either the white glove or one of the blue gloves). He discards this glove, so four gloves remain.

Now he reaches into the drawer again and chooses a glove at random from the four gloves that remain. What is the probability that this glove is blue?

3. Pair of dice. Roll a blue die and a red die. Given that the blue die has an odd value, which is the probability that the sum of the two dice is exactly 4?

4. Pair of dice. Roll a blue die and a red die. Given that the blue die has a value of 4 or smaller, what is the probability that the sum of the two dice is 7 or larger?

5. Coin flips and then dice. Claire flips a coin until she gets heads for the first time. Say it takes her n times. Then (afterwards) she rolls exactly n dice. What is the probability that none of the dice show the value 1?

(For instance, if it takes her 7 flips to get heads for the first time, then she rolls 7 dice.)

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Problem sets

Unit 3: Random Variables, Probability and Distributions

STAT/MA 41600

Practice Problems: September 10, 2014

1. Harmonicas. When ordering a new box of harmonicas, let X denote the time until the box arrives, and let Y denote the number of harmonicas that work properly.

Is X a continuous or discrete random variable? Why?

Is Y a continuous or discrete random variable? Why?

2. Choosing a page at random.

A student buys a brand new calculus textbook that has 1000 pages, each numbered with 3 digits, from 000 to 999. She randomly opens the book to a page and starts to read! Assume that any of the 1000 pages are equally likely to be chosen.

Let X be the page number of the chosen page. Thus, X is an integer-valued random variable between 0 and 999.

(a.) Find $P(X = 122)$.

(b.) Find $P(X = 977)$.

(c.) Find $P(X = -2)$.

(d.) Find $P(X = 1003)$.

(e.) When x is an integer between 0 and 999, find $P(X = x)$.

(f.) Find $P(X \leq 3)$.

(g.) Find $P(X \leq 122)$.

(h.) Find $P(12 \leq X \leq 17)$.

(i.) Find $P(X > 122)$.

(j.) Find $P(X = 15.73)$.

(k.) Find $P(X \leq 15.73)$.

3. Gloves. A matching pair of blue gloves, a matching pair of red gloves, and one lone white right-handed glove are in a drawer. The gloves are pulled out of the drawer, one at a time.

(a.) Suppose that a person is looking for the white glove. He repeatedly does the following: He pulls out a glove, checks the color, and if it is white, he stops. If it is not white, then he replaces the glove in the drawer and starts to check again, i.e., he reaches blindly into the drawer of 5 gloves. He continues to do this over and over until he finds the white glove, and then he stops. Let X be the number of draws that are necessary to find the white glove for the first time. For each positive integer j, find $P(X = j)$.

(b.) Now suppose that he searches for the white glove but, if he pulls a different colored glove from the drawer, he does not replace it. So he pulls out a glove, checks the color, and if it is white, he stops. If it is not white, then he *permanently discards* the glove and starts to check again, i.e., he reaches blindly into the drawer with the gloves that remain. He continues to do this over and over again until he finds the white glove, and then he stops. Let X be the number of draws that are necessary to find the white glove for the first time. For each integer j, with $1 \leq j \leq 5$, find $P(X = j)$.

4. Three dice. Roll three dice and let X denote the sum. For which values of j is $P(X = j)$ a strictly positive number?

5. Pick two cards. Pick two cards at random from a well-shuffled deck of 52 cards (pick them simultaneously, i.e., grab two cards at once—so they are not the same card!). There are 12 cards which are considered face cards (4 Jacks, 4 Queens, 4 Kings). Let X be the number of face cards that you get.

Find $P(X = 0)$.

Find $P(X = 1)$.

Find $P(X = 2)$.

STAT/MA 41600
Practice Problems: September 12, 2014

1. Butterflies. Alice, Bob, and Charlotte are looking for butterflies. They look in three separate parts of a field, so that their probabilities of success do not affect each other.

- Alice finds 1 butterfly with probability 17%, and otherwise does not find one.
- Bob finds 1 butterfly with probability 25%, and otherwise does not find one.
- Charlotte finds 1 butterfly with probability 45%, and otherwise does not find one.

Let X be the number of butterflies that they catch altogether.

Find the mass of X.

2. Appetizers. At a restaurant that sells appetizers:

- 8% of the appetizers cost \$1 each,
- 20% of the appetizers cost \$2 each,
- 32% of the appetizers cost \$3 each,
- 40% of the appetizers cost \$4 each.

An appetizer is chosen at random, and X is its price. Draw the CDF of X.

3. Wastebasket basketball. Chris tries to throw a ball of paper in the wastebasket behind his back (without looking). He estimates that his chance of success each time, regardless of the outcome of the other attempts, is $1/3$. Let X be the number of attempts required. If he is not successful within the first 5 attempts, then he quits, and he lets $X = 6$ in such a case.

Draw the mass of X.

Draw the CDF of X.

4. Two 4-sided dice. Consider some special 4-sided dice. Roll two of these dice and let X denote the sum.

Draw the mass of X.

Draw the CDF of X.

5. Pick two cards. Pick two cards at random from a well-shuffled deck of 52 cards (pick them simultaneously, i.e., grab two cards at once—so they are not the same card!). There are 12 cards which are considered face cards (4 Jacks, 4 Queens, 4 Kings). Let X be the number of face cards that you get.

Draw the CDF $F_X(x)$ of X.

STAT/MA 41600
Practice Problems: September 15, 2014

1. Butterflies. Alice, Bob, and Charlotte are looking for butterflies. They look in three separate parts of a field, so that their probabilities of success do not affect each other.

- Alice finds 1 butterfly with probability 17%, and otherwise does not find one.
- Bob finds 1 butterfly with probability 25%, and otherwise does not find one.
- Charlotte finds 1 butterfly with probability 45%, and otherwise does not find one.

Let X be the number of butterflies that they catch altogether. Let Y be the number of people who do not catch a butterfly.

Find the joint mass $p_{X,Y}(x, y)$ of X and Y.

2. Dependence/independence among dice rolls. A student rolls a die until the first "4" appears. Let X be the numbers of rolls required until (and including) this first "4." After this is completed, he begins rolling again until he gets a "3." Let Y be the number of rolls, after the first "4", up to (and including) the next "3." E.g., if the sequence of rolls is 213662341261613 then $X = 8$ and $Y = 7$. Are X and Y independent? Justify your answer.

3. Wastebasket basketball. Chris tries to throw a ball of paper in the wastebasket behind his back (without looking). He estimates that his chance of success each time, regardless of the outcome of the other attempts, is $1/3$. Let X be the number of attempts required. If he is not successful within the first 5 attempts, then he quits, and he lets $X = 6$ in such a case.

Let Y indicate whether he makes the basket successfully within the first three attempts. Thus $Y = 1$ if his first, second, or third attempt is successful, and $Y = 0$ otherwise.

Find the conditional mass of X given Y. You will need to list 12 values altogether, i.e., you need to compute $p_{X|Y}(x \mid y)$ for $1 \leq x \leq 6$ and $0 \leq y \leq 1$.

4. Two 4-sided dice. Consider some special 4-sided dice. Roll two of these dice. Let X denote the minimum of the two values that appear, and let Y denote the maximum of the two values that appear.

Find the joint mass $p_{X,Y}(x, y)$ of X and Y.

Find the joint CDF $F_{X,Y}(x, y)$ of X and Y. It suffices to give the values $F_{X,Y}(x, y)$ when x and y are integers between 1 and 4. You do not have to list any other values; only these 16 possibilities will suffice.

5. Pick two cards. Pick two cards at random from a well-shuffled deck of 52 cards (pick them simultaneously, i.e., grab two cards at once—so they are not the same card!). There are 12 cards which are considered face cards (4 Jacks, 4 Queens, 4 Kings). There are 4 cards with the value 10. Let X be the number of face cards in your hand; let Y be the number of 10's in your hand.

Are X and Y dependent or independent? Carefully justify your answer.

PurdueX: 416.1x
Probability: Basic Concepts & Discrete Random Variables

Problem sets

Unit 4: Expected Values

STAT/MA 41600
Practice Problems: September 17, 2014

1. Butterflies. Alice, Bob, and Charlotte are looking for butterflies. They look in three separate parts of a field, so that their probabilities of success do not affect each other.

- Alice finds 1 butterfly with probability 17%, and otherwise does not find one.
- Bob finds 1 butterfly with probability 25%, and otherwise does not find one.
- Charlotte finds 1 butterfly with probability 45%, and otherwise does not find one.

Let X be the number of butterflies that they catch altogether.

Find the expected value of X.

2. Dependence/independence among dice rolls. A student rolls a die until the first "4" appears. Let X be the numbers of rolls required until (and including) this first "4." After this is completed, he begins rolling again until he gets a "3." Let Y be the number of rolls, after the first "4", up to (and including) the next "3." E.g., if the sequence of rolls is 213662341261613 then $X = 8$ and $Y = 7$.

Find the expected value of X.

Find the expected value of Y.

3. Wastebasket basketball. Chris tries to throw a ball of paper in the wastebasket behind his back (without looking). He estimates that his chance of success each time, regardless of the outcome of the other attempts, is $1/3$. Let X be the number of attempts required. If he is not successful within the first 5 attempts, then he quits, and he lets $X = 6$ in such a case.

Find the expected value of X.

4. Two 4-sided dice. Consider some special 4-sided dice. Roll two of these dice. Let X denote the minimum of the two values that appear, and let Y denote the maximum of the two values that appear.

Find the expected value of X.

Find the expected value of Y.

5. Pick two cards. Pick two cards at random from a well-shuffled deck of 52 cards (pick them simultaneously, i.e., grab two cards at once—so they are not the same card!). There are 12 cards which are considered face cards (4 Jacks, 4 Queens, 4 Kings). There are 4 cards with the value 10. Let X be the number of face cards in your hand; let Y be the number of 10's in your hand.

Find the expected value of X.

Find the expected value of Y.

STAT/MA 41600
Practice Problems: September 19, 2014

1. Butterflies. Alice, Bob, and Charlotte are looking for butterflies. They look in three separate parts of a field, so that their probabilities of success do not affect each other.

- Alice finds 1 butterfly with probability 17%, and otherwise does not find one.
- Bob finds 1 butterfly with probability 25%, and otherwise does not find one.
- Charlotte finds 1 butterfly with probability 45%, and otherwise does not find one.

Let X be the number of butterflies that they catch altogether.

Write X as the sum of three indicator random variables, X_1, X_2, X_3 that indicate whether Alice, Bob, Charlotte (respectively) found a butterfly. Then $X = X_1 + X_2 + X_3$. Find the expected value of X by finding the expected value of the sum of the indicator random variables.

2. Dependence/independence among dice rolls. A student rolls a die until the first "4" appears. Let X be the numbers of rolls required until (and including) this first "4." After this is completed, he begins rolling again until he gets a "3." Let Y be the number of rolls, after the first "4", up to (and including) the next "3." E.g., if the sequence of rolls is 213662341261613 then $X = 8$ and $Y = 7$.

Let A_j be the event containing all outcomes in which "j or more rolls" are required to get the first "4." Let X_j indicate whether or not A_j occurs. Then $X = X_1 + X_2 + X_3 + \cdots$. Find the expected value of X by finding the expected value of the sum of the indicator random variables.

Let B_j be the event containing all outcomes in which "j or more rolls" are required, after the first "4", until he gets a "3". Let Y_j indicate whether or not B_j occurs. Then $Y = Y_1 + Y_2 + Y_3 + \cdots$. Find the expected value of Y by finding the expected value of the sum of the indicator random variables.

3. Wastebasket basketball. Chris tries to throw a ball of paper in the wastebasket behind his back (without looking). He estimates that his chance of success each time, regardless of the outcome of the other attempts, is $1/3$. Let X be the number of attempts required. If he is not successful within the first 5 attempts, then he quits, and he lets $X = 6$ in such a case.

Let A_j be the event containing all outcomes in which "j or more attempts" are required to get the basket. Let X_j indicate whether or not A_j occurs. Then $X = X_1 + X_2 + X_3 + X_4 + X_5 + X_6$. Find the expected value of X by finding the expected value of the sum of the indicator random variables.

4. Two 4-sided dice. Consider some special 4-sided dice. Roll two of these dice. Let X denote the minimum of the two values that appear, and let Y denote the maximum of the two values that appear.

Let A_j be the event containing all outcomes in which the minimum of the two dice is "j or greater." Let X_j indicate whether or not A_j occurs. Then $X = X_1 + X_2 + X_3 + X_4$. Find the expected value of X by finding the expected value of the sum of the indicator random variables.

5. Pick two cards. Pick two cards at random from a well-shuffled deck of 52 cards (pick them simultaneously, i.e., grab two cards at once—so they are not the same card!). There are 12 cards which are considered face cards (4 Jacks, 4 Queens, 4 Kings). There are 4 cards with the value 10. Let X be the number of face cards in your hand; let Y be the number of 10's in your hand.

Before looking at the cards, put one in your left hand and one in your right hand. Let X_1 and X_2 indicate, respectively, whether the cards in your left and right hands (respectively) are face cards. Then $X = X_1 + X_2$. Find the expected value of X by finding the expected value of the sum of the indicator random variables.

Before looking at the cards, put one in your left hand and one in your right hand. Let Y_1 and Y_2 indicate, respectively, whether the cards in your left and right hands (respectively) are 10's. Then $Y = Y_1 + Y_2$. Find the expected value of Y by finding the expected value of the sum of the indicator random variables.

STAT/MA 41600
Practice Problems: September 22, 2014

1a. Variance of an Indicator. Suppose event A occurs with probability p, and X is an indicator for A, i.e., $X = 1$ if A occurs, or $X = 0$ otherwise. We already know $\mathbb{E}(X) = p$. Find $\text{Var}(X)$.

1b. Butterflies. Alice, Bob, and Charlotte are looking for butterflies. They look in three separate parts of a field, so that their probabilities of success do not affect each other.

- Alice finds 1 butterfly with probability 17%, and otherwise does not find one.
- Bob finds 1 butterfly with probability 25%, and otherwise does not find one.
- Charlotte finds 1 butterfly with probability 45%, and otherwise does not find one.

Let X be the number of butterflies that they catch altogether. Find the variance of X.

2. Appetizers. At a restaurant that sells appetizers:

- 8% of the appetizers cost \$1 each,
- 20% of the appetizers cost \$2 each,
- 32% of the appetizers cost \$3 each,
- 40% of the appetizers cost \$4 each.

An appetizer is chosen at random, and X is its price. Each appetizer has 7% sales tax. So $Y = 1.07X$ is the amount paid on the bill (in dollars). Find the variance of Y.

3. Gloves. A matching pair of blue gloves, a matching pair of red gloves, and one lone white right-handed glove are in a drawer. The gloves are pulled out of the drawer, one at a time.

Suppose that a person is looking for the white glove. He repeatedly does the following: He pulls out a glove, checks the color, and if it is white, he stops. If it is not white, then he *permanently discards* the glove and starts to check again, i.e., he reaches blindly into the drawer with the gloves that remain. He continues to do this over and over again until he finds the white glove, and then he stops. Let X be the number of draws that are necessary to find the white glove for the first time.

Find $\mathbb{E}(X)$.

Find $\mathbb{E}(X^2)$.

Find $\text{Var}(X)$.

Find $\mathbb{E}(X^3)$.

4. Two 4-sided dice. Consider some special 4-sided dice. Roll two of these dice. Let X denote the minimum of the two values that appear, and let Y denote the maximum of the two values that appear.

Find the variance of X.

[Caution: If X_j is an indicator of whether the minimum of the two dice is "j or greater"—as in the previous homework—then $X = X_1 + X_2 + X_3 + X_4$, but the X_j's are dependent. So we cannot just sum the variances. We need to find the mass of X and then compute the expected value and variance by hand.]

5. Pick two cards. Pick two cards at random from a well-shuffled deck of 52 cards (pick them simultaneously, i.e., grab two cards at once—so they are not the same card!). There are 12 cards which are considered face cards (4 Jacks, 4 Queens, 4 Kings). There are 4 cards with the value 10. Let X be the number of face cards in your hand; let Y be the number of 10's in your hand.

Find the variance of X.

Find the variance of Y.

6. Design your own problem and solution. Create your own problem about either:

1. the expected value of a function of a discrete random variable, or
2. the variance of a discrete random variables.

Design your problem in such a way that you would find it enjoyable and also interesting (i.e., not completely trivial) if you found this problem in a probability book. Please provide the answer for your problem too.

PurdueX: 416.1x
Probability: Basic Concepts & Discrete Random Variables

Problem sets

Unit 5: Models of Discrete Random Variables I

STAT/MA 41600
Practice Problems: September 24, 2014

1. Winnings and Losing. Suppose that a person wins a game of chance with probability 0.40, and loses otherwise. If he wins, he earns 5 dollars, and if he loses, then he loses 4 dollars.

(a.) What is his expected gain or loss?

(b.) What is the variance of his gain or loss?

(c.) Find constants a, b such that if $X = 0$ when he loses and $X = 1$ when he wins, then $Y = aX + b$ is his earnings. [Hint: solve $5 = a(1) + b$ and $-4 = a(0) + b$. Also: X is a Bernoulli.] Verify your results above by finding $\mathbb{E}(Y)$ and $\mathrm{Var}(Y)$ with this method.

2. Telemarketers. One out of every eight calls to your house is a telemarketer. You will record whether the next call is a telemarketer.

(a.) What is the probability that the next time the phone rings, it will be a telemarketer?

(b.) If the phone calls are independent, what is the probability the 3rd call after dinner will be a telemarketer?

(c.) If you lose 30 seconds of your time every time you have to talk to a telemarketer, what is the expected amount of time you will lose on the next phone call?

(d.) What is the standard deviation of time you will lose on the next phone call?

3. Dating. You randomly call some of your friends who could be potential partners for a dance. You think that they all respond to your requests independently of each other, and you estimate that each one is 7% likely to accept your request. Let X_j indicate the event that you need to call j or more people until you find someone who accepted your invitation. So

$$X = X_1 + X_2 + X_3 + \cdots$$

is the total number of people that you need to call until someone accepts your invitation.

Find $\mathbb{E}(X)$.

4. Studying. Let X be the number of nights that you spend studying in a 30-day month. Assume that you study, on a given night, with probability .65, independent of the other nights. Write X as the sum of thirty indicators (i.e., as the sum of 30 Bernoulli random variables).

Find $\mathbb{E}(X)$.

Find $\text{Var}(X)$.

5. Shoes. Anne and Jane have shoes spread throughout the dorm room. Anne has 15 pairs of shoes; twenty percent of her shoe collection consists of sandals. Jane has 40 pairs of shoes; ten percent of her shoe collection consists of sandals.

(a.) A shoe is picked at random from the dorm room belonging to Anne and Jane; what is the probability that it is a sandal?

(b.) If a randomly-chosen shoe is chosen from the room (with all shoes equally likely to be chosen), what is the probability that it belongs to Anne?

(c.) If a randomly-chosen shoe is chosen from the room (with all shoes equally likely to be chosen), and upon examination this shoe is seen to be a sandal, what is the probability that it belongs to Anne?

STAT/MA 41600
More Practice Problems: September 24, 2014

1. Winnings and Losing. Suppose that a person wins a game of chance with probability 0.40, and loses otherwise. If he wins, he earns 5 dollars, and if he loses, then he loses 4 dollars. Assume that he plays ten games independently. Let X denote the number of games that he wins. [Hint: His gain or loss is $5X + (-4)(10 - X) = 9X - 40$, since he loses $10 - X$ games.]

(a.) What is his expected gain or loss (altogether) during the ten games?

(b.) What is the variance of his gain or loss (altogether) during the ten games?

(c.) What is the probability that he wins \$32 or more during the ten games?

2. Telemarketers. One out of every eight calls to your house is a telemarketer. Assume that the likelihood of telemarketers is independent from call to call. Let X denote the number of telemarketers during the next three calls.

(a.) What is the mass of X?

(b.) Draw a picture of the mass of X.

(c.) Draw a picture of the CDF of X.

3. Dating. You randomly call 20 of your friends who could be potential partners for a dance. You think that they all respond to your requests independently of each other, and you estimate that each one is 7% likely to accept your request.

(a.) Find the probability that at least 3 people would accept the invitation.

(b.) Find the expected number of people who would accept the invitation.

(c.) Find the variance of the number of people who would accept the invitation.

4. Dining Hall. Let X, Y, Z be (respectively) the number of nights that Alice, Bob, and Charlotte eat in the dining hall during a 7-day week. Assume that X, Y, Z are independent Binomial random variables that each have $n = 7$ and $p = .65$.

(a.) What is the distribution of $X + Y + Z$, i.e., the total number of meals eaten by these three people (altogether) during a week?

(b.) Find $\text{Var}(X + Y + Z)$.

5. Hearts. You draw seven cards, *without replacement*, from a shuffled, standard deck of 52 playing cards. Let X be the number of hearts that are selected.

(a.) What is the expected number of hearts? Why?

(b.) Is X a binomial random variable? Why or why not?

STAT/MA 41600
Practice Problems: September 26, 2014

1. Winnings and Losing. Suppose that a person wins a game of chance with probability 0.40, and loses otherwise. If he wins, he earns 5 dollars, and if he loses, then he loses 4 dollars. He plays the game until he wins for the first time, and then he stops. Assume that the games are independent of each other. Let X denote the number of games that he must play until (and including) his first win.

(a.) How many games does he expect to play until (and including) his first win?

(b.) What is the variance of the number of games he plays until (and including) his first win?

(c.) What is the probability that he plays 4 or more games altogether?

2. Winnings and Losing (continued). Continue to use the scenario from the previous problem. As before, let X denote the number of games that he must play until (and including) his first win.

(a.) Find a formula for his gain or loss, in terms of X. I.e., if Y denotes his gain or loss in dollars, write Y in terms of X.

(b.) What is his expected gain or loss (altogether) during the X games? I.e., what is $\mathbb{E}(Y)$?

(c.) What is the variance of his gain or loss (altogether) during the X games? I.e., what is $\mathrm{Var}(Y)$?

3. Telemarketers. One out of every eight calls to your house is a telemarketer. Assume that the likelihood of telemarketers is independent from call to call. Let X denote the number of callers until (and including) the next call by a telemarketer.

If n is a non-negative integer, what is $P(X > n)$?

4. Dating. You randomly call friends who could be potential partners for a dance. You think that they all respond to your requests independently of each other, and you estimate that each one is 7% likely to accept your request. Let X denote the number of phone calls that you make to successfully get a date.

(a.) Find the expected number of people you need to call, i.e., $\mathbb{E}(X)$.

(b.) Find the variance of the number of people you need to call, i.e., $\text{Var}(X)$.

(c.) Given that the first 3 people do not accept your invitation, let Y denote the additional number of people you need to call (Y does not include those first 3 people). I.e., Suppose $X > 3$ is given; then let $Y = X - 3$. Under these conditions, what is the mass of Y?

5. Hearts. You draw cards, one at a time, *with replacement* (i.e., placing them randomly back into the deck after they are drawn), from a shuffled, standard deck of 52 playing cards. Let X be the number of cards that are drawn to get the first heart that appears.

(a.) How many cards do you expect to draw, to see the first heart?

(b.) Now suppose that you draw five cards (again, with replacement), and none of them are hearts. How many additional cards (not including the first five) do you expect to draw to see the first heart?

STAT/MA 41600

Practice Problems (By Request, a special Harry Potter homework): September 29, 2014

1. Quidditch Training. Hermione is frustrated because she is extremely good at spells but she is struggling to learn how to fly on her broomstick. She repeatedly tries to fly on the broomstick. Assume that her trials are independent, and a trial succeeds with probability 0.15. She conducts trials until her 4th success, and then she stops. Let X denote the number of trials that are required until (and including) her 4th success with the broomstick.

(a.) What is the mass of X?

(b.) What is the probability that it takes Hermione exactly 12 trials until her 4th success?

(c.) What is the expected number of trials until Hermione's 4th success?

2. Horcruxes. Harry Potter needs to find 7 Horcruxes to defeat You-Know-Who. Harry makes repeated attempts to guess which objects are Horcruxes. Assume that his guesses about Horcruxes are independent. Each time he guesses about a Horcrux, he is correct only 1/3 of the time. Let X be the total number of times that he makes guesses until he finds all 7 Horcruxes.

(a.) What is the expected value of X?

(b.) What is the variance of X?

(c.) What is the probability that Harry finishes his quest to find all 7 Horcruxes on his 9th guess?

3. Mandrakes. According to legend, a mandrake plant screams when it is dug up, and it will kill anyone who hears the scream. So Professor Sprout asks the students to wear earmuffs as they are digging up the plants. Assume that the students each dig up their own mandrake, one at a time. Assume that the students dig up their plants independently, one at a time, in isolation. That way, only one student risks her/his own life at a time. Also assume that each student puts her/his earmuffs properly in place with probability 0.98 and therefore has a tragic death-by-mandrake-scream with probability 0.02.

The Ministry of Magic will allow Professor Sprout to continue this risky method of teaching until three tragedies occur. After three tragedies, they will immediately intervene and force Hogwarts to close. Let X denote the number of plants that are dug up until (and including) the third tragedy.

(a.) Find the expected number of Mandrakes that are dug up before Hogwarts is forced to close, i.e., the number of Mandrakes that are dug up until (and including) the third tragedy.

(b.) Find the variance of the number of Mandrakes that are dug up before Hogwarts is forced to close, i.e., the number of Mandrakes that are dug up until (and including) the third tragedy.

4. Divination. Professor Trelawney makes many predictions, but only 12% of them come true. The accuracy of the predictions are independent. Lavender makes a bet with Ron. Lavender gets 100 galleons when Professor Trelawney's prediction is correct. Lavender loses 15 galleons when Professor Trelawney is wrong. They play this game until Professor Trelawney's 5th success, and then they stop. Let X be the number of trials until (and including) Professor Trelawney's 5th success.

(a.) Find a formula for Lavender's earnings, Y, in terms of X. [Hint: When X trials are required, then 5 of them are successes and the other $X - 5$ are failures.]

(b.) Find Lavender's expected earnings, i.e., $\mathbb{E}(Y)$.

(c.) Find the variance of Lavender's earnings, i.e., $\mathrm{Var}(Y)$.

5. Spells. Suppose that Harry, Hermione, and Ron are each learning the Petrificus Totalus charm (i.e., the Full Body Bind). They each have probability of success 30% when they attempt the spell. Their attempts are each done completely independently (i.e., independently of their own earlier attempts; and also independently of the other people's attempts too). Harry tries the charm until his 5th success. Ron tries the charm until his 3rd success. Hermione, of course, wants to perfect the charm, so she tries the charm until her 20th success.

(a.) Let X denote the total number of attempts by all three children altogether (i.e., the sum of the number of their attempts). Find the expected value of X.

(b.) Find the variance of X.

PurdueX: 416.1x
Probability: Basic Concepts & Discrete Random Variables

Problem sets

Unit 6: Models of Discrete Random Variables II

STAT/MA 41600
Practice Problems: October 1, 2014

1. Hungry customers. At a certain hot dog stand, during the working day, the number of people who arrive to eat is Poisson, with an average of 1 person every 2 minutes.

a. What is the probability that exactly 3 people arrive during the next 10 minutes?

b. What is the probability that nobody arrives during the next 10 minutes?

c. What is the probability that at least 3 people arrive during the next 10 minutes?

2. Errors in Dr. Ward's book. Dr. Ward has carefully edited his book, but as all careful readers know, all books have some errors. In the first 250 pages, only 10 errors have been found altogether! (Hooray!) So it is reasonable to guess that the number of errors *per page* is Poisson, with an average of $10/250 = 0.04$ errors per page.

Suppose that the same low rate of errors continues in the second half of the book when it arrives, i.e., a Poisson number of errors, with an average of $10/250 = 0.04$ per page.

a. How many errors are expected in the next 100 pages of Dr. Ward's book?

b. What is the probability of exactly 5 errors in the next 100 pages of Dr. Ward's book?

3. Telemarketers. Suppose that, on average, 3 telemarketers call your house during a 7-day period.

a. What is the mass of the number of telemarketers calling your house during 1 day?

b. What is the probability that no telemarketers call your house on 1 given day?

c. What is the probability that exactly 2 telemarketers call your house on 1 given day?

4. Superfans. The number of Yankees fans shopping at a sports store, per hour, is Poisson with mean 8 per hour. The number of Red Sox fans shopping at the same store is Poisson with mean 6 per hour. Assume that the numbers of fans of the two types are independent. In particular, there is no person who is simultaneously a fan of both teams.

a. In a three hour period, how many Yankees and Red Sox fans do we expect altogether?

b. Find the probability that exactly 1 person enters the store during the next 20 minutes who likes the Yankees or Red Sox.

5. Shoppers. Suppose that the number of men who visit a website is Poisson, with mean 12 per minute, and the number of women who visit the same site is also Poisson, with mean 15 per minute. Assume that the number of men and women are independent.

a. During the next 10 seconds, what is the probability that 1 man and 2 women visit the site?

b. What is the variance of the total number of people who visit the site in the next 5 minutes?

STAT/MA 41600
Practice Problems: October 3, 2014

1. Hungry customers. At a certain restaurant, during the working day, there are 12 customers. Seven of them have pizza, and the other five have burgers. Suppose that a person is conducting a survey of three customers at the restaurant, and he conducts the survey without replacement, i.e., he does not talk to the same customer more than once. All possible selections of three people for the survey are equally likely.

Let X be the number of people in the survey who are eating pizza.

a. What is the mass of X?

b. Evaluate the mass at the four values where the mass is positive. (Make sure that your four numbers sum to 1.)

c. What is the average number of people in the survey who are eating pizza?

2. Harmonicas. Dr. Ward owns quite a few harmonicas. In particular, he has 7 "Deluxe" harmonicas and 12 "Crossover" harmonicas. Without looking at them, they have relatively similar shapes, so he does not notice a difference between them when he reaches into his harmonica container. Suppose that Dr. Ward grabs 8 harmonicas, without replacement, and all selections are equally likely.

a. How many Deluxe harmonicas does he expect to select?

b. What is the variance of the number of Deluxe harmonicas that he selects?

c. What is the probability that exactly 5 out of the 8 harmonicas are Deluxe?

3. Granola bars. I have 6 chocolate chunk granola bars, 10 raspberry granola bars, and 8 chocolate chip granola bars. I grab 3 without looking.

a. What is the probability that 2 are chocolate (either chunk or chip)?

b. What is the probability that strictly fewer than 2 are chocolate (either chunk or chip)?

c. What is the expected number of chocolate (either chunk or chip) granola bars out of the 3 that I grab?

4. Superfans. At Ross-Ade Stadium, there are 60,000 fans attending a football game. It is well known that only a few people at a Purdue football game will like Indiana University. Suppose that a person at the game likes Indiana University with probability $\frac{1}{10{,}000}$, independently of the other fans.

a. Give an exact formula for the probability that 8 of the people at the game like Indiana University. You do *not* have to evaluate the formula on your calculator.

b. Use a Poisson estimation for the probability above.

c. Use your calculator to evaluate the Poisson expression that you gave in part b.

5. Shoppers. During the holiday rush, there are 100,000 shoppers in a certain region. Each of these shoppers is extremely likely to make a purchase. Suppose that a person makes a purchase with probability 49,999/50,000 and declines to make a purchase with probability 1/50,000. Let X be the number of people who decline to make a purchase.

a. Give an exact formula for the probability that $P(X \leq 3)$. You do *not* have to evaluate the formula on your calculator.

b. Use a Poisson estimation for the probability above.

c. Use your calculator to evaluate the Poisson expression that you gave in part b.

STAT/MA 41600
Practice Problems: October 6, 2014

1. Hearts. In a game of chance, you are allowed to shuffle a standard deck of cards and then choose 3 cards randomly (without replacement). If two or more of them are hearts, then you win.

a. What is the probability of winning the game?

b. What is the expected number of hearts drawn?

2. Socks. In my sock drawer there are 21 white socks, 8 black socks, and 4 brown socks. (The socks are *not* folded into pairs.)

a. If I randomly pull out 6 socks to take with me on a trip, what is the probability that I pull out exactly 2 socks of each color?

b. What is the probability all the socks are the same color?

c. What is the probability that I pull out 2 socks of one color and 4 socks of a second color?

3. Married couples. A group of men and women sit in a circle. There are 20 chairs in the circle, and 10 pairs of married individuals. What is the expected number of men who are sitting directly across the room from their (respective) wives?

4. Ramen Noodles. There are 20 ramen noodles in a bag. There are 10 beef flavored, and the other 10 are chicken flavored.

a. What is the probability of getting at least one chicken and at least one beef flavor when you grab 3 packages randomly?

b. What is the probability that all three types are of the same flavor?

5. Picking letters at random. Five friends, named Albert, Bob, Charlie, Daniel, and Edward, each pick a letter from the alphabet on their own (i.e., independently), with all possible outcomes equally likely. For instance, they might pick (starting with Albert's choice) "M,D,P,Z,P".

a. What is the probability that they choose 5 distinct letters?

b. What is the probability that they choose 5 distinct letters *and that the letters are in increasing alphabetic order starting from Albert's choice and ending at Edward's choice*? For instance, starting with Albert's choice, "D,H,P,T,X" would be such an alphabetical ordering.