STUDENT VERSION

Combinatorics

Center for Science of Information
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Rule of Sum

Consider a scenario wtih lists of objects to choose from, but only one object overall can be chosen. How many possibilities exist? For example, let's say we have a list of ten numbers, from 1 through 10, and a list of fruits: an apple, a banana, and an orange. How many ways can we choose exactly one object from one of the two lists? We can choose either a single number or a single fruit, but not both.

Example 1

Let's look at some quick examples first. Let's say we have five red letters and three green letters. How many ways can we choose a letter? We can choose any of the five red letters or any of the three green letters, so there are $5 + 3 = 8$ different letters in total that we can choose. Note that the colors did not matter.

Example 2

We can repeat with five blue leters, four red letters, and three green letters, and although we can already see that the idea will be the same, let us consider the example nevertheless. How many ways can we choose a letter? We can choose any of the five blue letters, any of the four red letters, or any of the three green letters, so there are $3 + 4 + 5 = 12$ different letters in total that we can choose.

The Rule of Sum

In general, the Rule of Sum states that if we have a ways of doing something and b ways of doing another thing and we cannot do both at the same time then there are $a + b$ ways to choose one of the actions.

Question 1

Alice wants to order a one-topping pizza. The local pizza chain offers 3 different kinds of meat toppings, 4 different kinds of vegetables toppings, and 5 different kinds of cheese toppings. How many possible pizzas can Alice order?

Question 2

If there are a objects in S_1, b objects in S_2, and c objects in S_3, and the three sets S_1, S_2, and S_3 are pairwise disjoint, how many items are in their union?

Question 3

Bob wants to guess the first character of a password. He knows the character can either be a letter or a number. How many possible guesses does he have?

The Rule of Product

Consider a different scenario with lists of objects to choose from, but now one object is chosen from each list. How many possibilities exist?

Example 1

Let's look at some quick examples first. Let's say we have five blue letters and three red letters. How many ways can we choose two letters, one of which is red, and the other which is blue?

There are five options for the red letter. For each option of the red letter, there are three options for the blue letter, so there are $5 \cdot 3 = 15$ total ways we can choose two letters.

Example 2

Suppose we have five blue leters, four red letters, and three green letters. How many ways can we choose three letters, one of each color?

There are five options for the blue letter. For each option of the red letter, there are four options for the red letter. For each option of a red and blue letter, there are three options for the green letter, so there are $5 \cdot 4 \cdot 3 = 60$ total ways we can choose two letters.

Rule of Product

Consider a different scenario with lists of objects to choose from, but now one object is chosen from each list. For example, let's say we have a list of ten numbers, from 1 through 10, and a list of fruits: an apple, a banana, and an orange. How many ways can we choose exactly one object from one of the two lists? We must choose a number and a fruit. How many possibilities exist?

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The Rule of Product

If there are a ways of doing something and b ways of doing another thing, then there are ab ways of performing both actions.

It's important to note that the Rule of Product works as long as the events are independent, that is, the outcome of the first events do not affect the probabilities of the subsequent events.

Question 1

Suppose we draw a card from a standard 52-card deck, return it back to the deck, and draw another card. How many ways can we draw an Ace for the first card and a red card for the second card?

Question 2

Suppose we do this without returning the first card back to the deck. That is, we draw a card from a standard 52-card deck, and draw another card. Why is the number of ways can we draw an Ace for the first card and a red card for the second card different from the previous question?

Question 3

Suppose we have five red letters and three green letters. How many ways can we first choose a red letter and then choose a green letter? How many ways can we first choose a green letter and then a red letter? Why are the two answers the same?

the same because choosing a letter of one color does not effect the number of possibilities for the letters of the other color, as the actions are independent.

Permutations

Permutations are commonly thought of as rearrangements, permutations count the number of ways to rearrange objects, or a certain subset of objects. For exmple, suppose we have a list of integers from 1 through 10. How many ways can we rearrange the numbers? Let's look at some simpler examples first.

Example 1

How many ways can we choose a sequence of two letters from the letters ABC? For the first letter, we can choose any of the 3 options, A, B, or C. For the second letter, we can choose either of the 2 letters that we did not choose for the first letter. By the Rule of Product, we have $3 \cdot 2 = 6$ possibilities in total.

Example 2

How many ways are there to arrange 5 books on a shelf? There are 5 options for the first book, 4 options for the second book, 3 options for the third book, 2 options for the fourth book, and 1 option for the last book. By the Rule of Product, the number of ways is $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! = 120$. The exclamation point stands for factorial and is defined as following

$$k! = k \cdot (k-1) \cdot (k-2) \cdots 3 \cdot 2 \cdot 1$$

Example 3

How many ways are there to arrange 10 books on a shelf? By similar reasoning, there are 10! different rearrangements.

Example 4

How many ways are there to arrange 3 books on a shelf, if we have 5 books in total? There are 5 options for the first book, 4 options for the second book and 3 options for the third book, so by the Rule of Product, the number of ways is $5 \cdot 4 \cdot 3 = 60$.

Permutations

In general, if we have n objects, then there are $n \cdot (n-1) \cdots (n-k+1)$ ways to rearrange any k objects of the n objects. By the factorial notation, this is equivalent to

$$\frac{n!}{(n-k)!}$$

Question 1

How many rearrangements of the string ABC are there?

Question 2

How many ways are there to arrange n books on a shelf?

Question 3

How many ways are there to arrange m books on a shelf, if we have n books in total? $(n \geq m)$

Question 4

How many rearrangments of the word SOIHUB are there?

Combinations

Combinations enumerate the number of possibiliites in selecting objects from a list when the order of the selected objects does not matter. For exmple, if we were to count the number of possible poker hands, the order of the cards in your hand does not matter, so we want to use combinations, not permutations.

Example 1

If we want the number of possibilities of selecting 3 integers from a list of $1, 2, 3, \ldots, 10$, then we can count $\{7, 2, 8\}$ and $\{5, 7, 9\}$ as separate possibilities, but we cannot count $\{9, 7, 5\}$ as a possibility if we already counted

$\{5, 7, 9\}$. Doing so would result in a problem called overcounting, where we count some possibilities multiple times.

Example 2

How many ways can we select 2 letters from the set $\{A, B, C\}$? There are 3 options for the first letter, 2 options for the second letter, but we have overcounted. Each possibility of 2 letters has 2! possible rearrangements and we do not care about rearrangements, so we have overcounted by a factor of 2!. Thus, the number of possibilities is

$$\frac{3 \cdot 2}{2!} = 3.$$

Combinations

In general, if we seek the number of ways to choose k objects from a set of n objects, where $k \leq n$, the strategy is to count the number of ways to list k objects first, and then compensate for overcounting, because we do not actually care about the order of the k objects. From the permutations lesson, we know there are

$$\frac{n!}{(n-k)!}$$

ways of listing k objects from a set of n objects. Since there are k objects, there are $k!$ rearrangements. Hence, the total number of ways of selecting k objects from a set of n objets, also denoted $\binom{n}{k}$ is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Question 1

Calculate $\binom{6}{2}$

Question 2

Solve the example problem in the video. How many ways can we choose 3 integers from a set of 10 integers?

Question 3

How many ways are there to rearrange the sequence $BBBBBBDD$?

Question 4

Compute $\binom{6}{4}$ and $\binom{6}{2}$. Why are they the same?

n choose k

Choosing k objects from a set of n objects if the order of the chosen objects does not matter lends itself to $\binom{n}{k}$ different possibilities. From the previous lesson, we know that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Today, we will explore some identities involving the binomial coefficient.

Example 1

Each different possibility of choosing k objects from a set of n objects is the same as choosing the other $n-k$ objects to omit from the chosen set. Thus,

$$\binom{n}{k} = \binom{n}{n-k}.$$

Example 2

Since there is only one way to choose zero objects from the set, and there is only one way to choose all objects from the set,

$$\binom{n}{0} = \binom{n}{n} = 1.$$

Example 3 [Pascal's Identity]

Instead of choosing k objects immediately from the set of n objects, we may first consider a specific object. If we omit the specific object, we must choose the k objects from the remaining $n-1$ objects. Otherwise, if we include the specific object, we must choose the remaining $k-1$ objects from the remaining $n-1$ objects. Thus,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Question 1

Compute $\binom{7}{4}$.

Question 2

Compute $\binom{7}{4}$.

Question 3

Compute $\binom{7}{4} - \binom{7}{3}$.

Question 4

Compute $\binom{2014}{0}$.

Question 5

Compute $\binom{6}{2}$, $\binom{6}{3}$, $\binom{7}{3}$ and confirm $\binom{6}{2} + \binom{7}{2} = \binom{7}{3}$.

Binomial Coefficient

The binomial coefficient, pronounced n choose k is the number of ways to choose k objects from a set of n objects if the order of the chosen objects does not matter From the previous lessons, we know that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Why is it called the binomial coefficient? The binomial expansion is the expansion

$$(x+y)^n.$$

Example 1

What is the coefficient of x in the expansion of $(x+1)^3$? Since each term is a product of x's and 1's, we only get a term of x when we have a product of one x and two 1's. The number of ways to do this is $\binom{3}{1} = 3$.

Example 2

What is the coefficient of x^2 in the expansion of $(x+1)^3$? Since each term is a product of x's and 1's, we only get a term of x^2 when we have a product of two x's and one 1. The number of ways to do this is $\binom{3}{2} = 3$.

Binomial Coefficient

What is the coefficient of x^k in the expansion of $(x+1)^n$? Since each term is a product of x's and 1's, we only get a term of x^k when we have a product of k x's and $n-k$ 1's. The number of ways to do this is $\binom{n}{k}$, hence the name, binomial coefficient.

Question 1

What is the coefficient of $x^k y^{n-k}$ in the expansion of $(x+y)^n$?

Question 2

What is the coefficient of $x^2 y^3$ in the expansion of $(x+y)^6$?

Question 3

What is the coefficient of x^2 in the expansion of $(x+1)^5$?

Question 4

What is the coefficient of x^3 in the expansion of $(x+1)^7$?

Question 5

What is the coefficient of x^2 in the expansion of $(x+2)^5$?

Question 6

What is the coefficient of x^2 in the expansion of $(3x+2)^5$?

One-to-One Correspondence

If we can pair each object of one set to a different object of another set, such that all objects in both sets are paired, then we have a one-to-one correspondence between the two sets. When this occurs, we know the size or cardinality of the two sets are the same. This idea can be extended to infinite sets!

Example 1

If we have a classroom of 30 seats, we do not know how many students are enrolled in the class. Students could be absent or seats could be unoccupied.

Example 2

If we have a classroom of 30 seats, and all seats are occupied, we still do not know how many students are enrolled in the class. The class could be oversubscribed, and the extra students could be absent. Hence, we only know there are at least 30 students enrolled in the class.

Example 3

If we have a classroom of 30 seats, and all students are in attendance, we still do not know how many students are enrolled in the class. Seats could be unoccupied, so we only know there are at most 30 students enrolled in the class.

Example 4

If we have a classroom of 30 seats, all seats are occupied, and all students are in attendance, then we have a matching between each student and the seat which the student occupies. Hence, we have a one-to-one correspondence, and we know there are exactly 30 students enrolled in the class.

Example 5

If we have $A = \{1, 2, 3, 4, 5\}$ and $B = \{10, 8, 6, 4, 2\}$, then function f from A to B given by $f(x) = x + 1$ pairs $1 \in A$ and $2 \in B$. However, it pairs $2 \in A$ to 3, which is not in B. Hence, f is not a one-to-one correspondence between A and B. This does not necessarily mean A and B have different sizes. It just means the proposed f is not a one-to-one correspondence.

Example 6

If we have $A = \{1, 2, 3, 4, 5\}$ and $B = \{10, 8, 6, 4, 2\}$, then function f from A to B given by $f(x) = 2x$ pairs every element in A to a different element in B. Furthermore, all elements in A and B are paired, so then f is a one-to-one correspondence between A and B. Thus, A and B are the same size.

Question 1

Determine a one-to-one correspondence between $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 4, 9, 16, 25\}$.

Question 2

Determine a one-to-one correspondence between $A = \{1, 2, 3, 4, 5, \ldots\}$ and $B = \{2, 3, 4, 5, 6, \ldots\}$.

Question 3

Determine a one-to-one correspondence between $A = \{1, 2, 3, 4, 5, \ldots\}$ and $B = \{2, 4, 6, 8, 10, \ldots\}$.

Balls and Dividers

In this lesson, we will see an application of both one-to-one correspondences and binomial coefficients. As a review,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Example 1

How many ways are there to rearrange the sequence BBBBBBDD? One way we could choose to rearrange the sequence is by first looking at 8 slots for letters, choosing two of the slots to occupy with D's, and filling the rest of the slots with B's. There are $\binom{8}{2}$ ways to do this!

Example 2

Suppose we have six indistinguishable balls and three bins. How many ways can we assign balls to the three bins? By labeling the balls as letters, say B, then we have six B's. Since we have three bins, we can choose to have two dividers to divide the balls between the bins. The balls that appear before the first divider will appear in the first bin. The balls that appear between the first and second divider will appear in the second bin. Finally, the balls that appear after the second divider will appear in the third bin. By labeling the dividers with the letters D's, then we have six B's and two D's. By the previous example, there are $\binom{8}{2}$ rearrangements, so there are $\binom{8}{2}$ ways to distribute 6 balls among 3 bins. In this problem, we have a one-to-one correspondence between each rearrangement of BBBBBBDD and each distribution of six balls among three bins.

Balls and Dividers

In general, if we have n balls and k dividers, we will need n letters representing the balls, and $k-1$ letters representing the dividers. There are $\binom{n+k-1}{k-1}$ rerrangements of those letters, so by correspondence, there are $\binom{n+k-1}{k-1}$ distributions of n indistinguishable balls to k distinguishable bins.

Question 1

How many ways are there to rearrange the sequence with ten B's and three D's?

Question 2

How many ways are there to distribute ten balls among four bins?

Question 3

How many ways are there to distribute twelve balls among three bins?

Question 4

If x, y, z are non-negative integers, how many solutions are there to

$$x + y + z = 12.$$

Question 5

If w, x, y, z are non-negative integers, how many solutions are there to

$$w + x + y + z = 10.$$