Projections, Dissections and Bandwidth Expansion Mappings

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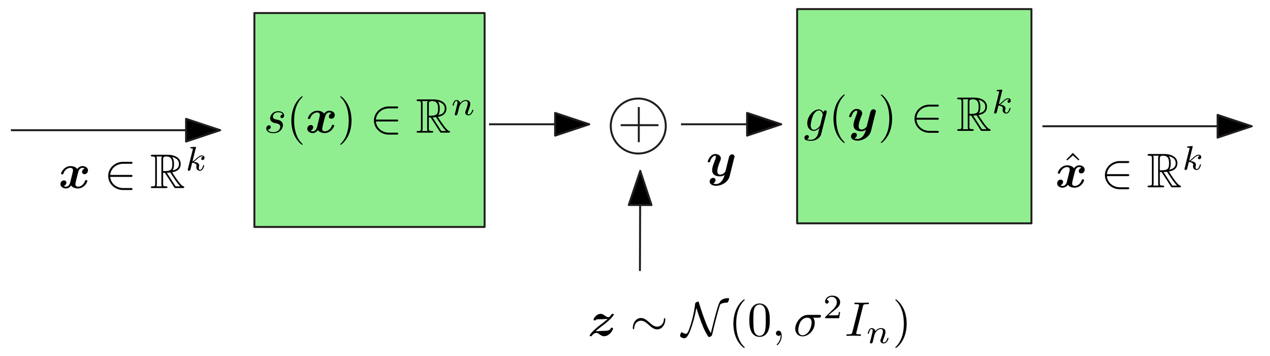
Communication system

- Vector $\boldsymbol{x} \in [0,1]^k$ drawn from a discrete-time continuous alphabet source transmitted over an $\boldsymbol{n}$-dimensional Gaussian channel



- Bandwidth expansion ratio n/k.
- Under constraint $E\left[\|s(x)\|^2\right] \leq P$, minimize MSE

$$\text{MSE} = \frac{1}{k} E\left[\|x - \hat{x}\|^2\right].$$

Information-Theoretical Limits

- Rate-Distortion Theory + Separation Principle:

$$D \geq \frac{1}{2\pi e (1 + \text{SNR})^{n/k}}$$

- Achievable with arbitrarily long (digital) block codes and infinite delay
- Question: How to design *explicit* and *efficient* analog mappings $s : [0,1]^k \to \mathbb{R}^n$ with asymptotically optimal behavior $\text{MSE} = \Theta(\text{SNR}^{-n/k})$?
- If $s(x)$ is linear, then $\text{MSE} = \Theta(\text{SNR}^{-1})$. Thus we must consider *non-linear* functions.

Cramér-Rao Bound and Low-Noise Approximation

The Cramér-Rao bound on the MSE for this model can be evaluated as

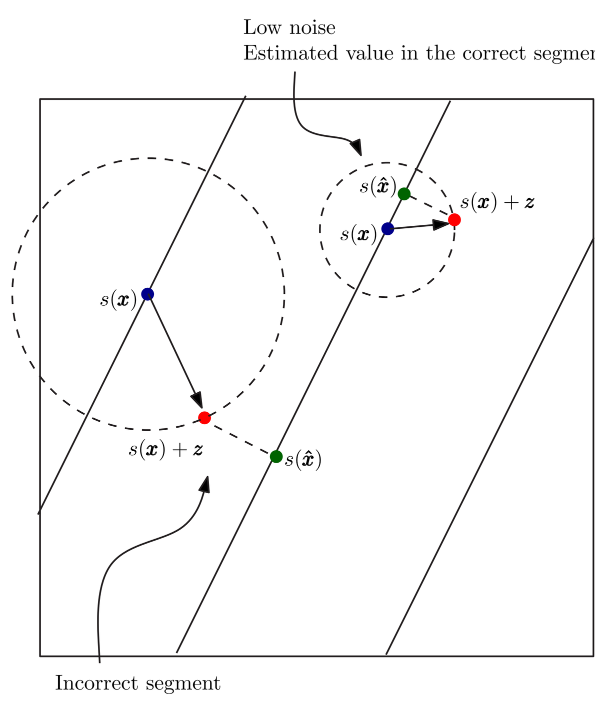
$$\frac{1}{k} E\left[\|x - \hat{x}\|^2\right] \geq \frac{\sigma^2}{k} \int_{[0,1)^k} \text{tr}((J(x)^t J(x))^{-1}) dx,$$

where $J(x)$ is the Jacobian of $s(x)$. In fact, if the noise is small (smaller than the distance between two "segments" of the curve), the MSE is well approximated by the CR-bound.

Threshold Effect

Design criteria:

- Maximize distance between "segments" of the locus $s([0,1]^k)$.
- "Stretch" the locus as much as possible/equally in each direction.

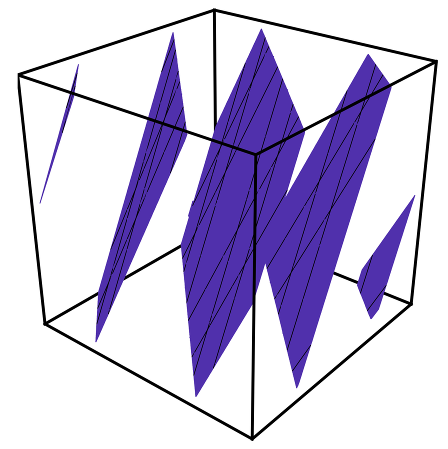


The mod-1 map

For $A \in \mathbb{Z}^{n \times k}$, we consider the piecewise linear map

$$s_1(x) = (A(x))_1 := Ax \pmod 1 = Ax - \lfloor Ax \rceil .$$

The map is injective iff A is a *primitive set of vectors* in $\mathbb{Z}^n$ (i.e., can be completed to a basis). Image consists of parallel "planes" inside the box $[-1/2, 1/2]^n$.



Distance between two segments:

$$\delta = \min_{n \in \mathbb{Z}^n, n \notin A^{\perp}} \min_{x \in \mathbb{R}^k} \|Ax - n\|$$

= the norm of the shortest vector in the lattice obtained by the projection of $\mathbb{Z}^n$ onto $A^{\perp}$. Tradeoff between minimum distance/determinant:

$$\rho = \frac{\alpha\delta}{2} = \frac{2\sqrt{3P}\Delta^{1/(n-k)}}{\sqrt{n}\det(A^t A)^{1/2(n-k)}}.$$

Analysis of the map

When there are no large errors:

$$\text{MSE} \approx \frac{\sigma^2 n \text{tr}(A^t A)^{-1}}{12kP},$$

but to meet the small error conditions we need ρ to be large $\Rightarrow$ $\det(A^t A)$ small. To achieve optimal exponent we need a family of matrices with:

1. (Injectivity) The columns of A are *primitive*.
2. (Minimum distance) The density of the projections of $\mathbb{Z}^n$ onto $A^{\perp}$ is bounded away from zero.
3. (MSE Exponent) $\text{tr}(A^t A)^{-1} = O(\det(A^t A)^{-1/k})$

(3.) is trivially satisfied if A is orthogonal. For some parameters $(k = n-2, n-1)$ constructions are possible. However, orthogonality + primitivity + good projections are hard to ensure simultaneously.

Ex: $(n-1)$ to n

Consider the matrix:

$$A_w = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ w & 1 & 0 & \cdots & 0 \\ 0 & w & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & w \end{pmatrix}.$$

Condition (1) and (2) are straightforward. Condition (3): $\det A_w^t A_w = \Theta(w^{2n-2})$ and $[(A_w^t A_w)^{-1}]_{ii} = \Theta(1/w^2)$. The associated mod-1 maps will have optimal exponent.

An Alternative Mapping: Modifying the support

By a matrix factorization, we can find Q and R, where $\det R = 1$ and columns of Q orthogonal, such that $A = QR$. Then the mapping:

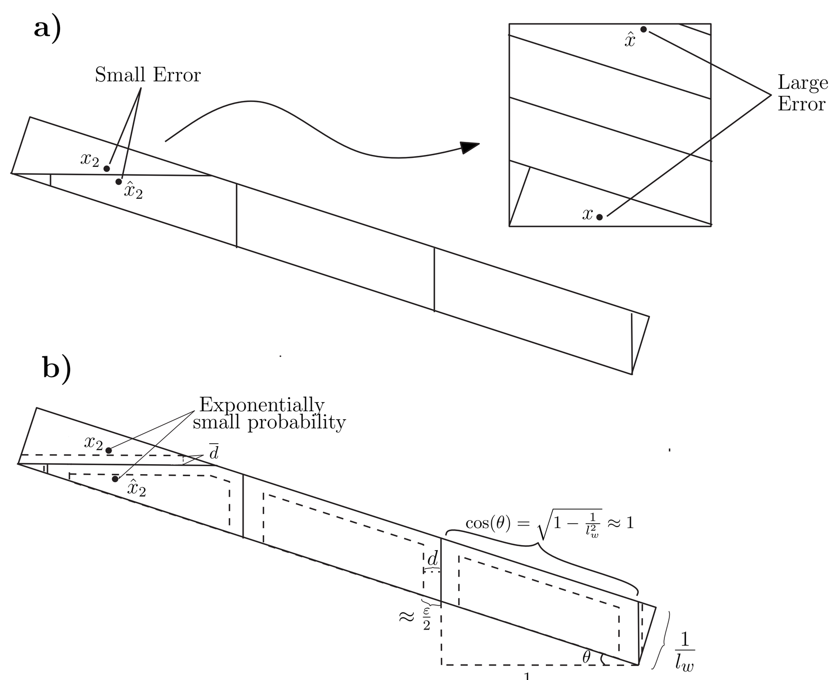
$$s_Q : \mathcal{S} \to \mathbb{R}^n$$
$$s_Q(x_2) = Qx_2 \pmod 1.$$

where $\mathcal{S} = R[0,1]^k$ yields an asymptotically optimal family (provided that A is chosen according to good projections). However the source is now $\mathcal{S} \neq [0,1]^k$ a parallelogram. If R^{-1} is applied to go back to $[0,1]^k$, it is possible that small errors will be magnified. To go back to the support $[0,1]^k$ we need an application that acts like an *isometry*.

Dissections of polyhedra

Idea: use s_Q and a bijection between the cube $[0,1]^k$ and $\mathcal{S}$ provided by a *dissection* to come back to the original support.

- *Dissect* $[0,1]^k$ and $\mathcal{S}$ into m non-overlapping polyhedra $T_1, T_2, \ldots, T_m$ and $\tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_m$ so that $[0,1]^k = \bigcup_{i=1}^m T_i$, $\mathcal{S} = \bigcup_{i=1}^m \tilde{T}_i$ and $\tilde{T}_i = \phi_i(T_i)$, where ϕ_i is an isometry.
  Define the map $s(x) = s_Q(\phi_i(x))$ if $x \in T_i$. Discontinuities can cause large errors. Solution: shrinking factor.



Proposition: For $k = 2$, there is a family of matrices and a proper choice of the shrinking factor such that $\text{MSE} = \Theta(\text{SNR}^{-n/2})$.
(very short) sketch of the proof: Choose a sequence of projections similar to [2] that exhibits optimal behavior after dissecting/before reassembling. If the shrinking factor is chosen properly, the degradation caused by the dissection technique is *exponentially small*, keeping the right behavior.

References

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2. N. J. A. Sloane, V. A. Vaishampayan, and S. I. R. Costa. The lifting construction: A general solution for the fat strut problem. In IEEE International Symposium on Information Theory Proceedings (ISIT), pp. 1037-1041, 2010.

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