Generating Dynamics by Varying an Entropic Framework

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Abstract

MaxEnt is a powerful tool for performing inference. As with any tool, its results are only as good as the model on which it operates. We will define a core model, which we will call the Entropic Framework (EF). The EF is comprised of a set of core assumptions and conditions which are general enough to include most Markovian systems. As we preform successive inferences, we are able to generate a dynamics, Entropic Dynamics (ED). If we vary this the EF, while maintaining the core assumptions and conditions, we will show that the EF is flexible enough to generate several forms of dynamics. To illustrate this we will be considering the trajectory of a point particle in configuration space. For such a system, we are able to generate diffusion, Quantum Mechanics, Classical Mechanics, etc.

Entropic Framework

As in any inference problem we start by defining our subject matter, assumptions, prior beliefs, and constraints. Together these will form framework of our inference, the Entropic Framework (EF).

Subject Matter:

Object of interest → ***particle***

Variable of interest → ***position***

The subject of inference will be a "particle", and we wish to make inference about particular values this particle takes, "position". Furthermore, the propose of this inference problem, will be to make predictions of the particle's future positions, trajectory.

Assumptions:

To this particle and position there are some operating assumptions which define the ontology of our universe of discourse.

1. The particle exists and has a definite position, x, but in general this position is not known. The particle lives in some space, $\mathcal{X}$ endowed with a metric, g_{ab}.

2. Change happens. A particle at a position, x, will "jump" to some new position, x'

3. There exist other "things" (extra variables) which may influence the particle. These variables take a value y and live in some space, $\mathcal{Y}$ with measure q(y). The value of these variables take of no consequence to the dynamics of the particle. It is only their uncertainties, p(y|x), that they contribute to the dynamics; more specifically through their entropy,

$$S(x) = -\int dy\, p(y\,|\,x) \log \frac{p(y\,|\,x)}{q(y)}$$

This entropy, is a measure on both, $\mathcal{X}$ space and the statistical manifold, $\mathcal{M}$. The manifold is introduced as a matter of convenience. It eliminates the need to directly deal with the y-variables, by assigning a p(y|x) to each x ε $\mathcal{M}$ (see Figure 1).

Since both the position variable and the extra variables, are relevant to the ontology, the joint space, $\mathcal{X} \times \mathcal{Y}$, is the proper arena to conduct the inference.

Priors:

A priori, we know nothing about the particle's position, the extra variables, nor any relationship between them. Therefore, we take the point of view of extreme ignorance,

4. Prior Probability, $Q(x',y'|x) \propto q(y) \times \sqrt{\det(g_{ab})}$

such that the prior is uniform; proportional to the volume elements of each space, respectively

Constraints:

We posit that there are preferred trajectories and constraints are introduced as a means for selecting these preferred trajectories. (see Figure 2)

We impose two constraints. The first maintains, that only the present has bearing on the future, Markov Process; the dynamics of p(y|x) are confined to the manifold, $\mathcal{M}$,

5. Markov Process: P(x',y'|x) = P(x'|x) × p(y'|x').

While the second constraint, allows for only small displacements, $\Delta\vec{x} = \vec{x}' - \vec{x}$, such that

6. Small Steps: $\langle\, l^2(x',x)\,\rangle = \langle\, g_{ab}\, \Delta x^a \Delta x^b \,\rangle = \lambda\,,$

where λ is some small but unspecified constant. This implies, a macroscopic displacement is the sum of many microscopic displacements.

EF Summary:

Briefly, the **Entropic Framework** states **there is exists an object**, the particle, and to it is some value of interest, position. This particle **undergoes change**, which is **influenced by other things**. The sum of these changes, form a trajectory which is not completely arbitrary. Rather this **trajectory is Markovian**, and comprised of many **small displacements** .

The Physical Example

To illustrate how a dynamics is generated by varying the EF, we will be considering the trajectory of a point particle living in a configuration space, with a given metric,

$$g_{ab} = \frac{\delta_{ab}}{\sigma^2}$$

where, δ is the flat euclidean metric & σ^2 is proportional to the mass of the particle

from MaxEnt towards an ED

Sparing much of the details and subtleties, MaxEnt is a tool used to process information, whereby one maximizes the entropy of the joint space, S[P,Q] by varying the transitional probability P(x'|x), subject to the prior, constraints, and normalization. The result of this process yields the transition probability

$$P(x'|x) = \frac{1}{\zeta(x,\alpha)}\, e^{S(x') - \frac{\alpha}{2\sigma^2}(\Delta\vec{x})^2}$$

where ζ is normalization constant and α is the lagrange multiplier that controls the size of the displacement. For large α, the displacements become small; these displacement are virtually isotropic with a slight bias along the direction of increasing entropy, S. So, displacement can be though of as the sum of two contributions, $\Delta\vec{x} = \Delta\bar{\vec{x}} + \Delta\vec{w}$; an expected displacement along the entropy gradient; and a random isotropic fluctuations, with

$$\langle\, \Delta\vec{x}\,\rangle = \Delta\bar{\vec{x}} \propto \alpha^{-1}\vec{\nabla} S(x) \;;\; \langle\, \Delta\vec{w}\,\rangle = 0 \;;\; \langle (\Delta\vec{w})^2 \rangle \propto \alpha^{-1}.$$

It is important to note, that as α→∞, the expected displacement does not die off as fast as the fluctuation, resulting in a continuous but non-differentiable trajectory, a Brownian motion.

Time:

Within inference, there is an inherited ordering of states of knowledge or beliefs. One begins with a prior state of beliefs, which are updated and become a posterior state of beliefs. This leads to an inherit asymmetry, between past, present, and future, such that time, in the entropic sense, always moves toward the future.

Time itself has been introduced as index which labels the state of knowledge of the particle's position at a particular instant in time. Here the state of knowledge of the particle position is codified in the probability distribution ρ(x,t).

For our propose, we wish time to have a Newtonian notion of time, where time is the linear successions of instants separated by a duration, Δt, (t' = t + Δt); where the duration is both temporally and spatially constant .

Time labels changes (in position). To this end we wish equate the size of change (in position) to the size of the duration.

$$\alpha = \frac{\tau}{\Delta t} = \mathit{Constant}$$

where τ is positive constant, such that τ << α. In ED, α is taken to be arbitrary large, therefore Δt is infinitesimally small.

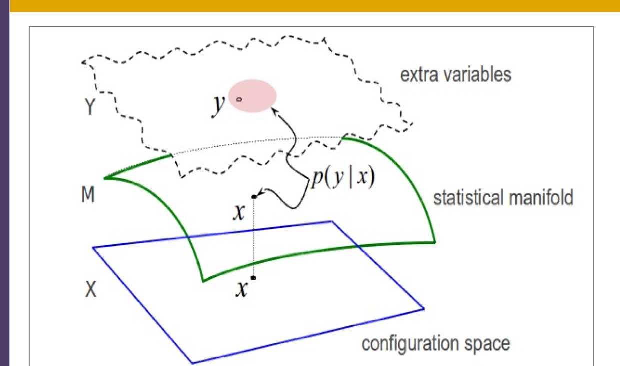

Figure 1:

These are the three spaces of the entropic framework. The configuration space ($\mathcal{X}$ space) is the 3-D space where the particle lives. Points in this space mark the particles position.

The extra variable space ($\mathcal{Y}$ space) is the residence of the extra variables. The true value, y, of these extra variables is unknown, but there exist a probability distribution (in pink) which codifies the uncertainty of these extra variables in terms of the position x, p(y|x).

The statistical manifold ($\mathcal{M}$ space) is constructed for convenience. At each point, x ε $\mathcal{M}$ (same x that labels the position in $\mathcal{X}$ space) there exist a probability distribution, p(y|x). So instead of directly working with the $\mathcal{Y}$ space, the $\mathcal{M}$ space can be used.

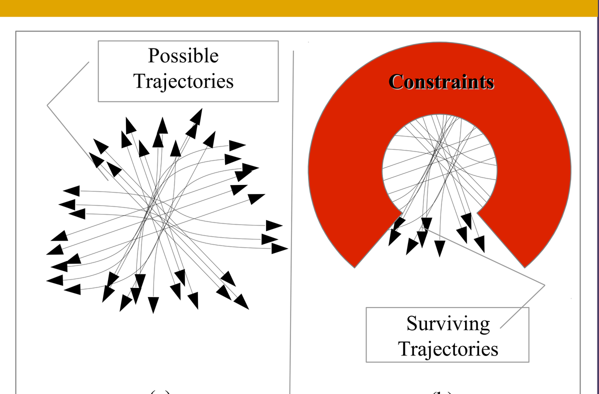

Figure 2:

(a) Possible trajectories, in the absence of constraints.

(b) When constraints are introduced (in red), one effectively limits the number of possible trajectories. However, not all trajectories are eliminated.

In the EF, we are not looking at one unique trajectory but an entire family of possible trajectories which are allowed by the constraints (as well as the prior).

Entropic Dynamics

We will consider different types of dynamics, which we can be generate by varying the Entropic Framework; such that the core assumptions are left intact.

EF Variation 1: Diffusion

Variation on the EF: **Do Nothing - Allow System to Evolve**

Resulting Entropic Dynamics: **Diffusion**

Allowing the system to evolve on it own, amounts to iterating the transitional probability, P(x'|x). Quantifying the accumulations of change results in a Fokker-Plank Equation,

$$\partial_t \rho = -\frac{\sigma^2}{\tau}\vec{\nabla}\cdot(\rho\,\vec{v})$$

where the current velocity, $\vec{v}$, is the sum of a drift and osmotic velocities,

$$\vec{v} = \vec{b} + \vec{u} = \frac{\sigma^2}{\tau}\vec{\nabla}\varphi \;;$$

with the drift velocity, $\vec{b} = \frac{\sigma^2}{\tau}\vec{\nabla} S(x)$; osmotic velocity $\vec{u} = -\frac{\sigma^2}{\tau}\vec{\nabla}\log\rho^{1/2}$;

and therefore **φ = S – ½ log ρ**.

The state of knowledge of the particles position, ρ, will diminish over time → diffusion. This is can be seen by inspecting the current velocity's components; the drift tends towards increasing entropy of the extra variables; while the osmotic velocity tend to move down the probability density gradient. Since the statistical manifold is static, given enough time, the osmotic tendencies will flatten ρ. This is a direct consequence of iterating an essentially isotropic process, P(x'|x).

For a slight variation on this theme, one could give the statistical manifold some externally prescribed time dependence. This would still lead to the diffusion of ρ; However, this would be like diffusion on a jiggling water bed → overtime the distribution will dampen out.

EF Variation 2: Quantum Mechanics

Variation on the EF: **Manifold "Waves" subject to Energy Conservation**

Resulting Entropic Dynamics: **The Schrödinger Equation (non-Relativistic QM)**

Quantum Mechanics requires two degrees of freedom, a density and phase. From diffusion, we have one degree of freedom, the density, ρ. If we allow the statistical manifold to participate in the dynamics, such that its evolution effects the ρ's and visa-versa; the phase, φ, becomes a true degree of freedom. To this end we require an energy,

$$E = \int dx \left(\tfrac{1}{2} m\,\vec{v}\cdot\vec{v} + \tfrac{1}{2}\mu\,\vec{u}\cdot\vec{u} + V(x,t) \right)$$

be conserved, $\partial_t \mathbf{E} = \mathbf{0}$. In a sense, this energy is the sum of kinetic (current and osmotic) and potential, V(x,t), energies. Where m and μ are the true and osmotic mass, respectively.

When energy conservation is implemented, we obtain a condition for the evolution of φ,

$$\hbar\dot{\varphi} + \frac{\hbar^2}{2m}\vec{\nabla}\varphi\cdot\vec{\nabla}\varphi + V - \frac{\mu\hbar^2}{4m^2}\frac{\nabla^2\rho}{\rho} = 0,$$

which is Quantum Hamilton-Jacobi Equation (QHJE). This in conjunction with the Fokker-Plank Equation completely describes the dynamics of the particle.

As a matter of convenience, we combine, ρ and φ into a single complex equation,

$$\Psi = \rho^{1/2} e^{i\varphi}$$

Inserting this into the QHJE; using the Fokker-Plank Eq.; and taking m = μ; we obtain,

$$i\hbar\dot{\Psi} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi$$

which is the Schrödinger Equation!

EF Variation 2a: Classical Mechanics

Variation on the EF: **Let $\hbar\varphi = S_{HJ}$, and take limit $\hbar \to 0$**

Resulting Entropic Dynamics: **Classical Mechanics (Classical Hamilton-Jacobi Eq.)**

Using the result of the QHJE as a starting point, we can recover Classical Mechanics, in the standard way, by taking the limit ħ → 0 (while keeping ħφ fixed). The result is the Classical Hamilton-Jacobi Equation,

$$\partial_t S_{HJ} + \frac{1}{2m}\left(\vec{\nabla} S_{HJ}\right)^2 + V = 0,$$

where S_{HJ} is the Action.

This is truly a classical picture, not only because we reproduce the Classical Hamilton-Jacobi Equation; but more importantly the fluctuations vanish!

$$\left|\langle (\Delta\vec{w})^2 \rangle\right| = \frac{\hbar}{m}\Delta t \quad \text{and as } \hbar \to 0 \quad \left|\langle (\Delta\vec{w})^2 \rangle\right| = 0$$

EF Variation 2b: Hybrid Mechanics

Variation on the EF: **Set μ = 0**

Resulting Entropic Dynamics: **Classical Hamilton-Jacobi Equation with Fluctuations**

One can also recover the Classical Hamilton-Jacobi Equation by setting the osmotic mass μ=0,

$$\partial_t S_{HJ} + \frac{1}{2m}\left(\vec{\nabla} S_{HJ}\right)^2 + V = 0,$$

However this is not a truly a classical picture, because the fluctuations do not vanish!

$$\left|\langle (\Delta\vec{w})^2 \rangle\right| = \frac{\hbar}{m}\Delta t \neq 0$$

This amounts to a particle fluctuating about a classical path. So this dynamics is neither fully classical nor fully quantum, rather a hybrid of the two.

EF Variation 3: Bohmian Mechanics

Variation on the EF: **Tune, τ & S → ∞ such that S/τ ≈ 1**

Resulting Entropic Dynamics: **deBroglie-Bohm Pilot Wave Interpretation of QM (dBB)**

Recall the lagrange multiplier α which is inversely proportional to the duration Δt with proportionality constant τ. We are free to adjust τ to whatever we like, such that τ << α.

When allowing τ to be arbitrarily large, while simultaneously adjusting S (to the same order), we recover the equations of the dBB.

When adjusting τ and S, the particle loses nearly all of its Brownian character: fluctuations die off faster than the drift; and to a good approximation the trajectory becomes differentiable and deterministic.

It is interesting to note, that the postulates (starting point) of the dBB, are simply derived in this form of ED; namely, the Schrödinger Equation and the Guidance Equation.

EF Variation 3: New Dynamics

Variation on the EF: $\partial_t S_T = 0$

Resulting Entropic Dynamics: **A Feedback-less Density**

We can construct a new form of dynamics completely separate from Quantum Theory. Starting with the ontology of the system, it being comprised of both position and extra variables, we can define the total entropy of the system, S_T, as

$$S_T = \underbrace{-\int dx\,\rho\log\rho}_{S[\rho],\ \text{entropy of x-vars}} + \int dx\,\rho\, S \quad .$$

If we impose the condition that S_T be static, what results is a phase, φ, whose evolution is independent of ρ,

$$\frac{3}{2}\frac{\hbar}{m}\nabla^2\varphi + \frac{\hbar}{m}\vec{\nabla}\varphi\cdot\vec{\nabla}\varphi + \dot{\varphi} = 0$$

However, the Fokker-Plank equation is still valid; therefore ρ's evolution is dependent on φ,

$$\frac{\hbar}{m}\vec{\nabla}\cdot\left(\rho\vec{\nabla}\varphi\right) + \dot{\rho} = 0$$

Whether this is of any Physical interest, requires further exploration.

Final Remarks

The Entropic Framework outlined here is robust yet malleable enough to generate several different forms of dynamics. Because of this, the EF may be a good starting point to tackle a variety of inference problems outside of Physics. To some approximation most systems where changes occur fit this Entropic Framework: there is an object that is changing; there may be other things that are contributing to the dynamics; initially, one has no idea about what this system is doing; the only thing that matters in predicting the future state is the present state; and seemly large changes are the accumulation many smaller changes.

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