Maximal Quaternion Orders Derived from $\{4g, 4g\}$ Tessellations

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Introduction

In a pioneer work in the context of designing communication systems on the hyperbolic plane [1], Brandani considers auto-dual tessellations $\{p, p\}$, which form an important subset of the tessellations $\{p, q\}$. Self-dual tessellations $\{4g, 4g\}$, where g denotes the genus of an orientable and compact surface, were considered in [2] for $g = 2, 3$ and extended in [3] for $g = 2^n, 3 \cdot 2^n, 5 \cdot 2^n$, where the corresponding arithmetic Fuchsian groups were derived from quaternion orders over quadratic extensions of the rationals. Since one of the main objectives is to design hyperbolic signal constellations it follows that each signal constellation has to be identified with a proper tessellation having an algebraic structure such as that of the quaternion order $\mathcal{O}$, or equivalently, a hyperbolic lattice.

Hyperbolic lattices $\mathcal{O}$ are the basic entities used in the design of signal constellations in the hyperbolic plane. Once the identification of the arithmetic Fuchsian group in a quaternion order is made, the next step is to determine the codewords of a code over a graph, or a signal constellation (quotient of an order by a proper ideal). However, in order for the algebraic labeling to be complete, it is necessary that the corresponding order be maximal. An order $\mathcal{M}$ in a quaternion algebra $\mathcal{A}$ is called *maximal* if $\mathcal{M}$ is not contained in any other order in $\mathcal{A}$, [4]. The main objective of this work is to describe the maximal orders derived from $\{4g, 4g\}$ tessellations.

Results

Let Γ_{4g} be a Fuchsian group, where $g = 2^n, 3.2^n, 5.2^n$ or $3.5.2^n$. Then Γ_{4g} is derived from a quaternion algebra $\mathcal{A} = (\theta, -1)_{\mathbb{Q}(\theta)}$, where θ is an algebraic number. Given p_m the minimal polynomial of degree m of θ, where $p_m(x) = x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0$, we show the discriminant of the maximal order $\mathcal{M}$ in $\mathcal{A}$ depends on the constant term, a_0, of the minimal polynomial, and on the basis of the maximal order in $\mathcal{A}$ that contains $\mathcal{O} = (\theta, -1)_{\mathbb{Z}(\theta)}$, as can be seen in the next results.

Theorem 1: *Given $g = 2^n$, the Fuchsian group Γ_{4g} is derived from a quaternion algebra $\mathcal{A} = (\theta, -1)_{\mathbb{Q}(\theta)}$, where $\theta = \sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}$ contains n radicals. The minimal polynomial of θ has degree $m = 2^n$ and the basis of the maximal order $\mathcal{M}$ in $\mathcal{A}$ is given by:*

$$\Big\{1, i, \frac{1}{2}\Big((\theta^{2^n-1} + \theta^{2^n-2} + \theta^{2^n-4} + \theta^{2^n-5} + \ldots + \theta^{2^{n-1}} + 1) + \theta^{2^n-1}i + j\Big),$$

$$\frac{-a_0}{4\theta}\Big(2 + (\theta^{2^n-1} + \theta^{2^n-2} + \theta^{2^n-4} + \theta^{2^n-5} + \ldots + \theta^{2^{n-1}} + 1) + \theta^{2^n-1}i + j\Big)\Big\}.$$

Furthermore, $(det(Tr(x_i \bar{x}_j))) = \frac{a_0}{2}$, that is, $d(\mathcal{M}) = \sqrt{\frac{a_0}{2}}$.

Theorem 2: *Given $g = 3.2^n$, the Fuchsian group Γ_{4g} is derived from a quaternion algebra $\mathcal{A} = (\theta, -1)_{\mathbb{Q}(\theta)}$, where $\theta = \sqrt{2+\sqrt{2+\ldots+\sqrt{3}}}$ contains $n+1$ radicals. The minimal polynomial has order $m = 2^{n+1}$ and the basis of the maximal order $\mathcal{M}$ in $\mathcal{A}$ is given by:*

$$\Big\{1, \frac{-a_0}{\theta}i, \frac{1}{2}\Big((\theta^{2^{n+1}-1} + \theta^{2^{n+1}-2} + \ldots + \theta + 1 - \theta^{2^n}) + (\theta^{2^{n+1}-1} +$$

$$\theta^{2^{n+1}-2} + \ldots + \theta + 1)i + j\Big), \frac{1}{2}\Big((\theta^{2^{n+1}-1} + \theta^{2^{n+1}-2} + \ldots + \theta +$$

$$1 - \theta^{2^n}) + (\theta^{2^{n+1}-1} + \theta^{2^{n+1}-2} + \ldots + \theta + 1)i + j\Big)\Big\}.$$

Furthermore, $(det(Tr(x_i \bar{x}_j))) = a_0$, that is, $d(\mathcal{M}) = \sqrt{a_0}$.

Theorem 3: *Given $g = 5.2^n$, the Fuchsian group Γ_{4g} is derived from a quaternion algebra $\mathcal{A} = (\theta, -1)_{\mathbb{Q}(\theta)}$, where $\theta = \sqrt{2+\sqrt{2+\ldots+\sqrt{2+\frac{\sqrt{10+2\sqrt{5}}}{2}}}}$ contains $n+2$ radicals. The minimal polynomial has order $m = 2^{n+2}$ and the basis of the maximal order $\mathcal{M}$ in $\mathcal{A}$ is given by:*

$$\Big\{1, \frac{-a_0}{\theta}i, \frac{1}{2}(\theta^{3.2^n} + j), \frac{1}{2}(\theta^{3.2^n}i + k)\Big\}.$$

Furthermore, $(det(Tr(x_i \bar{x}_j))) = a_0$, that is, $d(\mathcal{M}) = \sqrt{a_0}$.

Example: *Let $g = 10 = 5.2$. Then the associated quaternion order is $\mathcal{O} = (\theta, -1)_{\mathbb{Z}(\theta)}$, where $\theta = \sqrt{2+\frac{\sqrt{10+2\sqrt{5}}}{2}}$. The minimal polynomial is $p_8(x) = x^8 - 8x^6 + 19x^4 - 12x^2 + 1$, then the constant term is $a_0 = 1$. Thus, by the Theorem 3, the basis of the maximal order $\mathcal{M}$ in the quaternion algebra $\mathcal{A} = (\theta, -1)_{\mathbb{Q}(\theta)}$ is given by*

$$\Big\{1, \frac{-1}{\theta}i, \frac{1}{2}(\theta^6 + j), \frac{1}{2}(\theta^6 i + k)\Big\},$$

and the discriminant is $d(\mathcal{M}) = 1 = d(\mathcal{A})$.

Conclusions

We have derived the maximal orders associated with the tessellations $\{4g, 4g\}$, from which codes over hyperbolic lattices with a complete algebraic labeling may be obtained.

Acknowledgements

References

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