Entropy Power Inequality and Mrs. Gerber's Lemma for Abelian Groups of Order 2^n

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Introduction

• The Entropy Power Inequality (EPI) proposed by Shannon in 1948 [1] is an inequality in the so called "entropy power" of valued random variables. Entropy power of a random variable **X** is defined as

$$N(\mathbf{X}) = \frac{1}{2\pi e} e^{\frac{2}{n} h(\mathbf{X})}$$

The EPI states that for independent random variables **X** and **Y**,

$$N(\mathbf{X}) + N(\mathbf{Y}) \leq N(\mathbf{X} + \mathbf{Y})$$

and equality holds iff **X** and **Y** are Gaussian with proportional covariance matrices.

• The EPI has been generalized in a variety of ways. Notable generalizations include Costa's [2] concavity of entropy power and Zamir and Feder's [3] generalization to linear transformations of random variables.

• Several attempts have been made to obtain discrete versions of the EPI. Shamai and Wyner [4] used a result called Mrs. Gerber's Lemma (MGL) proved by Wyner and Ziv [5] to obtain a binary analog of EPI.

• We approach the discrete EPI problem in a different, straight-forward way and attempt to get a version of the EPI for finite abelian groups.

Our Interpretation

• Even though the EPI is thought of as an inequality in terms of "entropy power", it is essentially a sharp lower bound on *h*(**X**+**Y**) in terms of *h*(**X**) and *h*(**Y**). Thus, for any abelian group *G* with the binary operation +, we can examine the function

$$f_G(x, y) = \min_{H(X)=x, H(Y)=y} H(X+Y)$$

where X,Y are *G* valued random variables.

• For real valued random variables,

$$f_{\mathbb{R}}(x, y) = \frac{1}{2} \log \left(e^{2x} + e^{2y}\right)$$

• For $\mathbb{Z}_2$ we have

$$f_{\mathbb{Z}_2}(x, y) = H(H^{-1}(x) \star H^{-1}(y))$$

Key Observation

Wyner and Ziv's MGL can be stated in terms of

MGL: $f_{\mathbb{Z}_2}(x, y)$ *is convex in x for a fixed y, and vice versa.*

The above convexity property holds even for $f_{\mathbb{R}}$!

It is natural to make the conjecture:

Conjecture (Generalized MGL): *For a finite abelian group G, is convex in x for a fixed y, and vice versa.*

Proof Technique and Results

• We prove that $\frac{\partial f_{\mathbb{Z}_2}}{\partial x}$ decreases along lines through the origin

• Using this we prove that $f_{\mathbb{Z}_2}$ is concave along such lines, and also that the pair of partial derivatives $\left(\frac{\partial f_{\mathbb{Z}_2}}{\partial x}, \frac{\partial f_{\mathbb{Z}_2}}{\partial y}\right)$ uniquely determines a point (x, y).

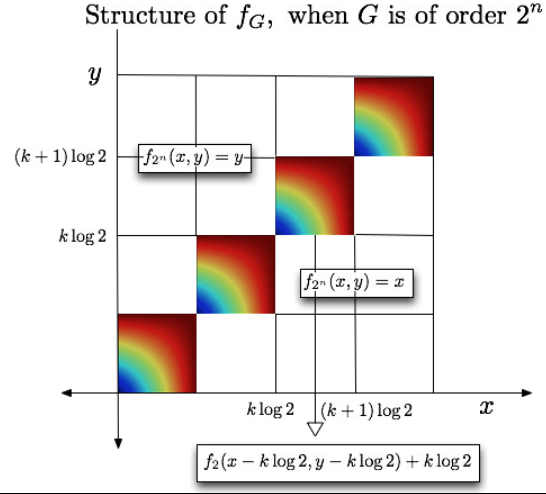
• We use the above lemmas and explicitly determine $f_{\mathbb{Z}_4}$. Using this as the base case for induction, we make a guess and prove the explicit form of $f_{\mathbb{Z}_{2^n}}$.

• It is then easily checked that the form of $f_{\mathbb{Z}_{2^n}}$ indeed does satisfy the conjecture.

• Now that we have the form of f_G for cyclic groups, we extend it to arbitrary groups of order 2^n by using the fundamental theorem of abelian groups and inducting over the number of cyclic groups being direct summed.

• We also describe those distributions where minimum entropy is attained for such groups, these can be thought of as analogs to Gaussians in the real case.

Structure of f_G, when G is of order 2^n



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